ON CEGRELL'S CLASSES OF m-SUBHARMONIC FUNCTIONS

LU HOANG CHINH

Abstract. We study finite energy classes of m-subharmonic functions which generalise Cegrell's classes for plurisubharmonic functions [5]. We then use a variational method inspired by [4] to solve the degenerate complex Hessian equation $(dd^c\varphi)^m \wedge \beta^{n-m} = \mu$, where μ is a positive singular Radon measure.

Contents

1. Introduction

Let Ω be a bounded domain of $\mathbb{C}^n$ and m be an integer such that $1 \leq m \leq n$. We consider complex m-Hessian equations of the form

$$(dd^c\varphi)^m \wedge \beta^{n-m} = \mu, \tag{1.1}$$

where $\beta := dd^c|z|^2$ is the standard Kähler form in $\mathbb{C}^n$ and μ is a positive Radon measure.

The border cases $m = 1$ and $m = n$ correspond to the Laplace equation which is a classical subject and the complex Monge-Ampère equation which was studied intensively recent years by many authors.

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As complex Monge-Ampère equation, the complex m-Hessian equation is elliptic when restricted to m-subharmonic functions. Nevertheless, due to a lack of positivity the complex m-Hessian equation is much more difficult to handle.

The complex m-Hessian equation was first studied by Li [19]. He used the well-known continuity method to solve the non-degenerate Dirichlet problem for equation (1.1) (where the data is smooth and we seek for smooth solutions) in strongly m-pseudoconvex domains. One of its degenerate counterparts was studied by Błocki [3]. More precisely, he solved the homogeneous equation with continuous boundary data and developed first steps of a potential theory for this equation. Recently, Abdullaev and Sadullaev also considered m-polar sets and m-capacity for m-subharmonic functions [26]. When the right-hand side μ has density in $L^p(\Omega)$ ($p > n/m$) Dinew and Kołodziej proved that given a continuous boundary data, the Dirichlet problem of equation (1.1) has a unique continuous solution [7]. The Hölder regularity of the solution has been recently studied by Nguyen Ngoc Cuong [23]. He also showed how to lead to solutions from subsolutions [22]. A viscosity approach to this equation has been developed in [21] which generalise results in [29] and [9].

The correspondent complex m-Hessian equation on compact Kähler manifolds has been studied by many authors including the author of this paper. It has the following form

$$(\omega + dd^c\varphi)^m \wedge \omega^{n-m} = f\omega^n, \tag{1.2}$$

where (X, ω) is a compact Kähler manifold of dimension n and $1 \leq m \leq n$. This is a generalisation of the well-known Calabi-Yau equation [27]. Due to a lack of positivity the proof of Yau's Theorem can not be copied.

When $f > 0$ is a smooth function satisfying the compatibility condition $\int_X f\omega^n = \int_X \omega^n$, Dinew and Kołodziej recently proved that (1.2) has a unique (up to additive constant) smooth solution. This had been known to hold when ω has non-negative bisectional curvature (see [13], [15]) or even restrictive classes of Kähler manifolds [17].

When $0 \leq f \in L^p(X, \omega^n)$ for some $p > n/m$, Dinew and Kołodziej recently proved that (eq: heq kahler) admits a unique continuous weak solution. The result also holds when the right-hand side $f = f(x, \varphi)$ depends on the unknown [20].

The *real* Hessian equation is a classical subject which was studied intensively recent years. The reader can find a survey for this in [28]. It has been warned that *real* and *complex* Hessian equation are very different and direct adaptation often fails [7].

In this paper, we develop next steps of a potential theory for the complex m-Hessian equations in $\mathbb{C}^n$. We follow Cegrell [5] to define classes of finite energy of m-subharmonic functions. As any one who is working on this subject may easily recognise, our results are easy generalisations of classical results of Cegrell.

The paper is organised as follow. In section 2, we recall basic facts about m-subharmonic functions and the complex m-Hessian operators. The proof of these facts can be found in [22], [23], [26] or in [16] modulo some easy modifications. In section 4, we study finite energy classes of m-subharmonic functions inspired by [5, 6]. In section 5, we use a variational approach inspired by [4] (see also [1]) to solve equation 1.1 with a quite singular right-hand side.

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2. Preliminaries

In the whole paper, β denotes the standard Kähler form in $\mathbb{C}^n$, and Ω is a bounded m-hyperconvex domain in $\mathbb{C}^n$, i.e. there exists a continuous m-subharmonic function $\varphi : \Omega \to \mathbb{R}^-$ such that $\{\varphi < c\} \Subset \Omega$, for every $c < 0$. The results in this section can be proved similarly as in the case of plurisubharmonic functions (see [16], [18], [22, 23], [26]).

2.1. **Elementary symmetric functions.** Let S_k, $k = 1, ..., n$ be the k-elementary symmetric function, that is, for $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{R}^n$,

$$S_k(\lambda) = \sum_{1\leq i_1<i_2<...<i_k\leq n} \lambda_{i_1}\lambda_{i_2}...\lambda_{i_k}.$$

For convenience, we set $S_0(\lambda) = 1$ and $S_k(\lambda) = 0$ if $k > n$ or $k < 0$. We have the following identity

$$(\lambda_1 + t)...(\lambda_n + t) = \sum_{k=0}^{n} S_k(\lambda)t^{n-k}, \quad t \in \mathbb{R}.$$

We denote Γ_k the closure of the connected component of $\{S_k(\lambda) > 0\}$ containing $(1, ..., 1)$. We can show that (see [10])

$$\Gamma_k = \{\lambda \in \mathbb{R}^n \ / \ S_k(\lambda_1 + t, ..., \lambda_n + t) \geq 0, \ \forall t \geq 0\}.$$

Thus it follows from the identity

$$S_m(\lambda_1 + t, ..., \lambda_n + t) = \sum_{k=0}^{m} \binom{n-k}{m-k} S_k(\lambda)t^{m-k}, \quad t \in \mathbb{R}$$

that

$$\Gamma_k := \{\lambda \in \mathbb{R}^n \ / \ S_j(\lambda) \geq 0, \quad \forall 1 \leq j \leq k\}.$$

We have an obvious inclusion $\Gamma_n \subset ... \subset \Gamma_1$. By Gårding [10] the set Γ_k is a convex cone in $\mathbb{R}^n$ and $S_k^{1/k}$ is concave on Γ_k.

We denote by $\mathcal{H}$ the vector space (over $\mathbb{R}$) of complex Hermitian $n \times n$ matrices. For $A \in \mathcal{H}$ we let $\lambda(A) = (\lambda_1, ..., \lambda_n)$ denote the eigenvalues of A. We set

$$\widetilde{S}_k(A) = S_k(\lambda(A)).$$

From the identity

$$\det(A + tI) = \sum_{k=0}^{n} \widetilde{S}_k(A)t^{n-k}, \quad t \in \mathbb{R}$$

it follows that the function $\widetilde{S}_k$ can be seen as the sum of all principal minors of order k,

$$\widetilde{S}_k(A) = \sum_{|I|=k} A_{II}.$$

Thus $\widetilde{S}_k$ is a homogeneous polynomial of order k on $\mathcal{H}$ which is hyperbolic with respect to the identity matrix I (that is for every $A \in \widetilde{S}$ the equation $\widetilde{S}_k(A+tI) = 0$ has n real roots; see [10]). As in [10] (see also [3]) we define the cone

$$\widetilde{\Gamma}_k := \{A \in \mathcal{H} \ / \ \widetilde{S}_k(A+tI) \geq 0, \forall t \geq 0\}.$$

We have

$$\widetilde{\Gamma}_k := \{A \in \mathcal{H} \ / \ \lambda(A) \in \Gamma_k\}.$$

It follows from [10] that the cone $\widetilde{\Gamma}_k$ is convex and the function $\widetilde{S}_k^{1/k}$ is concave on $\widetilde{\Gamma}_k$.

2.2. **m-subharmonic functions and the Hessian operator.** We associate real (1,1)-forms α in $\mathbb{C}^n$ with Hermitian matrices $[a_{j\bar{k}}]$ by

$$\alpha = \frac{i}{\pi} \sum_{j,k} a_{j\bar{k}} dz_j \wedge d\bar{z}_k.$$

Then the canonical Kähler form β is associated with the identity matrix I. It is easy to see that

$$\binom{n}{k} \alpha^k \wedge \beta^{n-k} = \widetilde{S}_k(A)\beta^n.$$

Definition 2.1. Let α be a real $(1,1)$-form on Ω. We say that α is m-positive at a given point $P \in \Omega$ if at this point we have

$$\alpha^j \wedge \beta^{n-j} \geq 0, \quad \forall j = 1, ..., k.$$

α is called k-positive if it is k-positive at any point of Ω.

Let T be a current of bidegree $(n-k, n-k)$ $(k \leq m)$. Then T is called m-positive if

$$\alpha_1 \wedge ... \wedge \alpha_k \wedge T \geq 0,$$

for all m-positive $(1,1)$-forms $\alpha_1, ..., \alpha_k$.

Definition 2.2. A function $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ is called m-subharmonic if it is subharmonic and

$$dd^c u \wedge \alpha_1 \wedge ... \wedge \alpha_{m-1} \wedge \beta^{n-m} \geq 0,$$

for every m-positive (1,1) forms $\alpha_1, ..., \alpha_{m-1}$. The class of all m-subharmonic functions in Ω will be denoted by $\mathcal{SH}_m(\Omega)$.

Proposition 2.3. [3] *(i) If u is $\mathcal{C}^2$ smooth then u is m-subharmonic if and only if the form $dd^c u$ is m-positive at every point in Ω.*

(ii) If $u, v \in \mathcal{SH}_m(\Omega)$ then $\lambda u + \mu v \in \mathcal{SH}_m(\Omega), \forall \lambda, \mu > 0$.

(iii) If u is m-subharmonic in Ω then the standard regularization $u \star \chi_\epsilon$ are also m-subharmonic in $\Omega_\epsilon := \{x \in \Omega \ / \ d(x, \partial\Omega) > \epsilon\}$.

(iv) If $(u_l) \subset \mathcal{SH}_m(\Omega)$ is locally uniformly bounded from above then $(\sup u_l)^\star \in \mathcal{SH}_m(\Omega)$, where $v^\star$ is the upper semi continuous regularization of v.

(v) $PSH = \mathcal{P}_n \subset ... \subset \mathcal{P}_1 = SH$.

(vi) Let $\emptyset \neq U \subset \Omega$ be a proper open subset such that $\partial U \cap \Omega$ is reltively compact in Ω. If $u \in \mathcal{P}_m(\Omega)$, $v \in \mathcal{SH}_m(U)$ and $\limsup_{x \to y} v(x) \leq u(y)$ for each $y \in \partial U \cap \Omega$ then the function w, defined by

$$w = \begin{cases} u \ \text{on} \ \Omega \setminus U \\ \max(u,v) \ \text{on} \ U \end{cases},$$

is m-subharmonic in Ω.

For locally bounded m-subharmonic functions $u_1, ..., u_p$ ($p \leq m$) we can inductively define a closed m-positive current (following Bedford and Taylor [2]).

Lemma 2.4. *Let $u_1, ..., u_k (k \leq m)$ be locally bounded m-subharmonic functions in Ω and let T be a closed m-positive current of bidegree $(n-p, n-p)$ $(p \geq k)$. Then we can define inductively a closed m-positive current*

$$dd^c u_1 \wedge dd^c u_2 \wedge ... \wedge dd^c u_k \wedge T,$$

and the product is symmetric, i.e.

$$dd^c u_1 \wedge dd^c u_2 \wedge ... \wedge dd^c u_p \wedge T = dd^c u_{\sigma(1)} \wedge dd^c u_{\sigma(2)} \wedge ... \wedge dd^c u_{\sigma(k)} \wedge T,$$

for every permutation $\sigma : \{1, ..., k\} \to \{1, ..., k\}$.

In particular, the Hessian measure of $\varphi \in \mathcal{SH}_m(\Omega) \cap L^\infty_{loc}$ is defined to be

$$H_m(u) = (dd^c u)^m \wedge \beta^{n-m}.$$

Proof. See [26]. □

Proposition 2.5. *Let T be a closed m-positive current of bidegree $(n-1, n-1)$ on Ω. Let u, v be bounded m-subharmonic functions in Ω such that $u, v \leq 0$ and*

$$\lim_{z \to \partial\Omega} u(z) = 0.$$

Then one has

$$\int_\Omega v dd^c u \wedge T \leq \int_\Omega u dd^c v \wedge T. \tag{2.1}$$

If we assume moreover that

$$\lim_{z \to \partial\Omega} v(z) = 0,$$

then in (2.1), the equality holds

$$\int_\Omega v dd^c u \wedge T = \int_\Omega u dd^c v \wedge T.$$

Proof. The proof is an easy adaptation from the classical case (see [22, 23]). □

Proposition 2.6 (Chern-Levine-Nirenberg inequality)**.** *Let $K \Subset D \Subset \Omega$ with K compact, D open. Then there exists $A > 0$ such that*

$$\|dd^c u_1 \wedge ... \wedge dd^c u_k \wedge T\|_K \leq A.\|u_1\|_{L^\infty(D)} ... \|u_k\|_{L^\infty(D)} \|T\|_D$$

and

$$\|v dd^c u_1 \wedge ... \wedge dd^c u_k \wedge T\|_K \leq A.\|u_1\|_{L^\infty(D)} ... \|u_k\|_{L^\infty(D)} \int_D |v| T \wedge \beta^p,$$

for each m-subharmonic function v which is integrable with respect to a closed m-positive current T of bidegree $(n-p, n-p)$ $(p \geq k)$, and all locally bounded m-subharmonic functions $u_1, ..., u_k$.

Theorem 2.7. *Let $(u_0^j), ...(u_k^j)$ be decreasing sequences of m-subharmonic functions in Ω converging to $u_0, ..., u_k \in \mathcal{SH}_m(\Omega) \cap L^\infty_{loc}$ respectively. Let T be a closed m-positive current of bidegree $(n-p, n-p)$ $(p \geq k)$ on Ω. Then*

$$u_0^j . dd^c u_1^j \wedge ... \wedge dd^c u_k^j \wedge T \rightharpoonup u_0 . dd^c u_1 \wedge ... \wedge dd^c u_k \wedge T,$$

weakly in the sense of currents.

2.3. **The relative m-capacity.** One of the most important properties of m-subharmonic functions is the quasicontinuity. Every m-subharmonic function is continuous outside an arbitrarily small open subset. The m-capacity is used to measure the smallness of these sets. As in the case of plurisubharmonic functions, the convergence in L^1_{loc} is not sufficient to deduce the convergence of the Hessian operator. We will see that the convergence in m-capacity is "good enough" for our purpose.

Definition 2.8. Let $E \subset \Omega$ be a Borel subset. The m-capacity of E with respect to Ω is defined to be

$$Cap_m(E, \Omega) := \sup \Big\{ \int_E H_m(\varphi) \ / \ \varphi \in \mathcal{SH}_m(\Omega), 0 \leq \varphi \leq 1 \Big\}.$$

Remark 2.9. By Chern-Levine-Nirenberg inequality, $Cap_m(K, \Omega)$ is finite for any compact $K \Subset \Omega$. Moreover, $Cap_m(E, \Omega) \geq C \int_E d\lambda_n$, where C is a constant depending on n and the diameter of Ω and λ_n is the volume form in $\mathbb{C}^n$.

The m-capacity shares the same elementary properties as Cap_{BT}.

Proposition 2.10. *i) $Cap_m(E_1, \Omega) \leq Cap_m(E_2, \Omega)$ if $E_1 \subset E_2$,*
ii) $Cap_m(E, \Omega) = \lim_{j\to\infty} Cap_m(E_j, \Omega)$ if $E_j \uparrow E$,
iii) $Cap_m(E, \Omega) \leq \sum Cap_m(E_j, \Omega)$ for $E = \cup E_j$.

Proposition 2.11. *Let $K \Subset U \Subset \Omega$. Then there exists a constant C depending on these sets such that for any $u \in \mathcal{SH}_m(\Omega)$, $u < 0$, we have*

$$Cap_m(K \cap \{u < -j\}, \Omega) \leq \frac{C}{j} \|u\|_{L^1(U)}.$$

Proof. Fix $v \in \mathcal{SH}_m(\Omega)$ with $-1 \leq v < 0$. Then by CLN inequalities

$$\int_{K\cap\{u<-j\}} (dd^c v)^m \wedge w^{n-m} \leq \frac{1}{j} \int_K |u| (dd^c v)^m \wedge w^{n-m} \leq \frac{C}{j} \|u\|_{L^1(U)}.$$

□

Definition 2.12. A sequence u_j of functions defined in Ω converges in m-capacity to u if for any $t > 0$ and $K \Subset \Omega$ one has

$$\lim_{j\to\infty} Cap_m(K \cap \{|u - u_j| > t\}) = 0.$$

The following results can be proved by repeating the arguments in [18].

Proposition 2.13. *A sequence $u_j \in \mathcal{SH}_m(\Omega) \cap L^\infty_{loc}$ with $u_j \downarrow u$ in Ω converges to $u \in \mathcal{SH}_m(\Omega) \cap L^\infty_{loc}$ with respect to m-capacity.*

Theorem 2.14. *For an m-subharmonic function u defined in Ω and a positive number ϵ one can find an open set $U \subset \Omega$ with $Cap_m(U, \Omega) < \epsilon$ and such that u restricted to $\Omega \setminus U$ is continuous.*

Theorem 2.15. *Let $\{u_k^j\}_{j=1}^\infty$ be a locally uniformly bounded sequence of m-subharmonic functions in Ω for $k = 1, 2, ..., N \leq m$ and let $u_k^j \uparrow u_k \in \mathcal{SH}_m(\Omega) \cap L^\infty_{loc}$ almost everywhere as $j \to \infty$ for $k = 1, 2, ..., N$. Then*

$$dd^c u_1^j \wedge ... \wedge dd^c u_N^j \wedge \beta^{n-m} \rightharpoonup dd^c u_1 \wedge ... \wedge dd^c u_N \wedge \beta^{n-m}.$$

Corollary 2.16. *Let u_j be a monotone sequence of locally bounded m-subharmonic functions in Ω converging almost everywhere to $u \in \mathcal{SH}_m(\Omega) \cap L^\infty_{loc}$ and f_j a monotone sequence of locally bounded m-quasicontinuous functions converging almost everywhere to a locally bounded quasicontinuous function f. Then*

$$f_j(dd^c u_j)^m \wedge \beta^{n-m} \rightharpoonup f(dd^c u)^m \wedge \beta^{n-m}.$$

Proposition 2.17 (Maximum principle)**.** *Let $\Omega \Subset \mathbb{C}^n$, $u, v \in \mathcal{SH}_m(\Omega) \cap L^\infty_{loc}$. Then*

$$\mathbb{1}_{\{u>v\}} H_m(\max(u, v)) = \mathbb{1}_{\{u>v\}} H_m(u).$$

Corollary 2.18 (Comparison principle)**.** *Let $u, v \in \mathcal{SH}_m(\Omega) \cap L^\infty_{loc}$ such that*

$$\liminf_{z \to \partial\Omega} (u(z) - v(z)) \geq 0.$$

Then

$$\int_{\{u<v\}} (dd^c v)^m \wedge \beta^{n-m} \leq \int_{\{u<v\}} (dd^c u)^m \wedge \beta^{n-m}.$$

Corollary 2.19. *Let Ω be a bounded domain in $\mathbb{C}^n$, and $u, v \in \mathcal{SH}_m(\Omega) \cap L^\infty_{loc}$ are such that $u \leq v$ on $\partial\Omega$ and $H_m(u) \geq H_m(v)$. Then $u \leq v$ in Ω.*

2.4. **The relative m-extremal function.**

Definition 2.20. For a subset E of a domain $\Omega \subset \mathbb{C}^n$ we define the relative m-extremal function by

$$u_{m,E,\Omega} = u_{m,E} := \sup\{u \in \mathcal{SH}_m(\Omega) \ / \ u < 0, \text{ and } u \leq -1 \text{ on } E\}.$$

It is easy to see that $u^\star_{m,E,\Omega}$ is m-subharmonic in Ω. If there is no confusion we will use the notation u_E and $u^\star_E$ instead of $u_{m,E,\Omega}$ and $u^\star_{m,E,\Omega}$.

Proposition 2.21. *i) If $E_1 \subset E_2$ then $u_{E_2} \leq u_{E_1}$.*
ii) If $E \subset \Omega_1 \subset \Omega_2$ then $u_{E,\Omega_2} \leq u_{E,\Omega_1}$.
iii) If $K_j \downarrow K$, with K_j compact in Ω then $(\lim u^\star_{K_j,\Omega})^\star = u^\star_{K,\Omega}$.

Lemma 2.22. *Let $0 < r < R$ and set $a = \dfrac{n}{m} > 1$. The m-extremal function $u := u_{m,\overline{B}(r),B(R)}$ is given by*

$$u(z) = \max\Big(\frac{R^{2-2a} - \|z\|^{2-2a}}{r^{2-2a} - R^{2-2a}}, -1\Big).$$

Proposition 2.23. *If E is a relatively compact subset of Ω, then at any point $w \in \partial\Omega$ one has*

$$\lim_{z \to w} u_{m,E,\Omega}(z) = 0.$$

Proposition 2.24. *Let $K \subset \Omega$ be a compact subset which is the union of closed balls, then $u^\star_K = u_K$ is continuous. In particular, if $K \subset \Omega$ is an arbitrary compact set and $\epsilon <$ dist$(K, \partial\Omega)$, then u_{K_ϵ} is continuous, where*

$$K_\epsilon = \{z \in \Omega \ / \ dist(z, K) \leq \epsilon\}.$$

Definition 2.25. An m-subharmonic function $u : \Omega \to \mathbb{R}$ is m-maximal if for any open relatively compact subset $G \Subset \Omega$ and any upper semicontinuous function v in $\overline{G}$, $v \in \mathcal{SH}_m(G)$ and $v \leq u$ on ∂G then $v \leq u$ in G.

It follows from the comparison principle that every locally bounded m-subharmonic function u satisfying $H_m(u) = 0$ in Ω is m-maximal. Conversely any m-maximal function u in Ω satisfies $H_m(u) = 0$.

Proposition 2.26. [3] *Let Ω be an open subset of $\mathbb{C}^n$ and u is an m-maximal function in Ω. Then $H_m(u) = 0$.*

Proposition 2.27. *If $K \subset \Omega$ is compact, then $u^*_{m,K,\Omega}$ is m-maximal in $\Omega \setminus K$.*

Theorem 2.28. *Let $K \subset \Omega$ be a compact subset, set $u = u_{m,K,\Omega}$ the relative m-extremal function. Then*

$$Cap_m(K, \Omega) = \int_K H_m(u^\star).$$

Moreover, if $u^\star > -1$ on K then $Cap_m(K, \Omega) = 0$.

Corollary 2.29. *If U is an open relatively compact subset of Ω, then*

$$Cap_m(U, \Omega) = \int_\Omega H_m(u_{m,U,\Omega}).$$

In the following example we compute the m-capacity of the balls.

Example 2.30. For every $0 < r < R$ we have

$$Cap_m(B(r), B(R)) = \frac{2^n(n-m)}{m.n!(r^{2-2a} - R^{2-2a})^m}.$$

Definition 2.31. Let Ω be an open set in $\mathbb{C}^n$, and let $\mathcal{U} \subset \mathcal{SH}_m(\Omega)$ be a family of functions which is locally bounded from above. Define

$$u(z) = \sup\{v(z) \ / \ v \in \mathcal{U}\}.$$

Sets of the form $\mathcal{N} = \{z \in \Omega \ / \ u(z) < u^\star(z)\}$ and all their subsets are called m-negligible.

Definition 2.32. A set $E \subset \mathbb{C}^n$ is called m-polar if for any $z \in E$ there exist a neighbourhood V of z and $v \in \mathcal{SH}_m(V)$ such that $E \cap V \subset \{v = -\infty\}$. If $E \subset \{v = -\infty\}$ for some $v \in \mathcal{SH}_m(\mathbb{C}^n)$ then E is called globally m-polar.

Theorem 2.33. *A subset E of $\mathbb{C}^n$ is m-negligible if and only if it is m-polar.*

Proposition 2.34. *If Ω is an open subset of $\mathbb{C}^n$ and $u \in \mathcal{SH}_m(\Omega) \cap L^\infty_{loc}(\Omega)$ then for any m-polar subset $E \subset \Omega$ one has*

$$\int_E H_m(u) = 0.$$

Theorem 2.35. *E is an m-polar set if and only if there exists $u \in \mathcal{SH}_m(\mathbb{C}^n)$ such that $u \equiv -\infty$ on E.*

3. Finite energy classes of Cegrell type

In this section we study finite energy classes of m-subharmonic functions in m-hyperconvex domains. They are generalizations of Cegrell's classes [5, 6] for plurisubharmonic functions.

3.1. **Definitions and properties.** In pluripotential theory one of the most important steps is to regularize singular plurisubharmonic functions. It can be easily done locally by convolution with a smooth kernel. The following theorem explain how to do it globally in m-hyperconvex domain. Let $\mathcal{SH}_m^-(\Omega)$ denote the class of non-positive functions in $\mathcal{SH}_m(\Omega)$.

Theorem 3.1. *For each $\varphi \in \mathcal{SH}_m^-(\Omega)$ there exists a sequence (φ_j) of m-sh functions verifying the following conditions:*
(i) φ_j is continuous on $\bar{\Omega}$ and $\varphi_j \equiv 0$ on $\partial\Omega$;
(ii) each $H_m(\varphi_j)$ has finite mass, i.e $\int_\Omega H_m(\varphi_j) < +\infty$;
(iii) $\varphi_j \downarrow \varphi$ on Ω.

Proof. If B is a closed ball in Ω then by Proposition 2.24 the m-extremal function $u := u_{m,B,\Omega}$ is continuous on $\bar{\Omega}$ and supp$H_m(u) \Subset \Omega$. We follow [6, Theorem 2.1].

Take a decreasing sequence of positive numbers (r_j) such that

$$0 < r_j < dist\Big(\Big\{u(z) < -\frac{1}{2j^2}\Big\}, \partial\Omega\Big).$$

Let ψ_j be the regularization sequence of φ by convolution with a smooth kernel. This sequence is well-defined on

$$\Omega_j := \{z \in \Omega : dist(z, \partial\Omega) > 1/j\}.$$

Set

$$v_p(z) = \begin{cases} \max\big(\psi_{r_p}(z) - \dfrac{1}{p}, p.u(z)\big) & \text{if } \ z \in \Omega_{r_p} \\ p.u(z) \ \text{ if } \ z \in \Omega \setminus \Omega_{r_p}, \end{cases}$$

and

$$\varphi_j := \sup_{p \geq j} v_p.$$

Without difficulty, we can check that for each p, $v_p \in \mathcal{SH}_m(\Omega) \cap \mathcal{C}(\bar{\Omega})$ and $v_p \equiv 0$ on $\partial\Omega$. We then deduce that φ_j is lower semicontinuous for every j. Furthermore by setting

$$\varphi_j^k := \max\{v_j, ..., v_{k-1}, v_k + \frac{1}{k}\}, k > j,$$

we see that $\varphi_j^k \downarrow \varphi_j$ when $k \to +\infty$. Since each φ_j^k is continuous, we obtain the upper semicontinuity hence continuity of φ_j. It is clear that $\varphi_j \downarrow \varphi$ on Ω. $\square$

Definition 3.2. We let $\mathcal{E}_m^0(\Omega)$ denote the class of bounded functions in $\mathcal{SH}_m^-(\Omega)$ such that $\lim\limits_{z\to\partial\Omega} \varphi(z) = 0$ and $\int_\Omega H_m(\varphi) < +\infty$.

For each $p > 0$, $\mathcal{E}_m^p(\Omega)$ denote the class of functions $\varphi \in \mathcal{SH}_m(\Omega)$ such that there exists a decreasing sequence $(\varphi_j) \subset \mathcal{E}_m^0(\Omega)$ satisfying
(i) $\lim_j \varphi_j = \varphi$, on Ω and
(ii) $\sup_j \int_\Omega (-\varphi_j)^p H_m(\varphi_j) < +\infty$.

If we require moreover that $\sup_j \int_\Omega H_m(\varphi_j) < +\infty$ then, by definition, φ belongs to $\mathcal{F}_m^p(\Omega)$.

Definition 3.3. A function $u \in \mathcal{SH}_m^-(\Omega)$ belongs to $\mathcal{E}_m(\Omega)$ if for each $z_0 \in \Omega$, there exists an open neighborhood $U \subset \Omega$ of z_0 and a decreasing sequence $(h_j) \subset \mathcal{E}_m^0(\Omega)$ such that $h_j \downarrow u$ in U and

$$\sup_j \int_\Omega H_m(h_j) < +\infty.$$

We let $\mathcal{F}_m(\Omega)$ denote the class of functions $u \in \mathcal{SH}_m^-(\Omega)$ such that there exists a sequence $(u_j) \subset \mathcal{E}_m^0(\Omega)$ decreases to u in Ω and

$$\sup_j \int_\Omega H_m(u_j) < +\infty.$$

Remark 3.4. $\mathcal{SH}_m^-(\Omega) \cap L^\infty_{loc} \subset \mathcal{E}_m(\Omega)$. In fact, let $u \in \mathcal{SH}_m^-(\Omega)$ and $z_0 \in B(z_0, r) \Subset \Omega$. Consider the function $h := h_{m,B,\Omega}$. We know that $h \in \mathcal{E}_m^0(\Omega)$ and $h \equiv -1$ in B. For each $A > 0$ big enough we have $\max(u, Ah) \in \mathcal{E}_m^0(\Omega)$ and $\max(u, Ah) = u$ in B.

Theorem 3.5. *The class $\mathcal{E}_m(\Omega)$ is the biggest subclass of $\mathcal{SH}_m^-(\Omega)$ which satisfies the following conditions:*
(i) if $u \in \mathcal{E}_m(\Omega), v \in \mathcal{SH}_m^-(\Omega)$ then $\max(u, v) \in \mathcal{E}_m(\Omega)$.
(ii) if $u \in \mathcal{E}_m(\Omega), \varphi_j \in \mathcal{SH}_m^-(\Omega) \cap L^\infty_{loc}, u_j \downarrow u$, then $H_m(u_j)$ is weakly convergent.

Proof. It is clear that $\mathcal{E}_m(\Omega)$ satisfies the condition (i). Suppose that $u \in \mathcal{E}_m(\Omega), u_j \in \mathcal{SH}_m^-(\Omega) \cap L^\infty_{loc}, u_j \downarrow u$. Fix χ a test function with compact support $K \Subset \Omega$, and $h \in \mathcal{E}_m^0(\Omega)$. For each j we can find n_j such that $u_j \geq n_j.h$ in a neighborhood of K. By setting $\varphi_j := \max(u_j, n_j.h) \in \mathcal{E}_m^0(\Omega)$ we see that $\varphi_j \downarrow u \in \mathcal{E}_m(\Omega)$, and $H_m(\varphi_j)$ is weakly convergent to $H_m(u)$ by definition of $\mathcal{E}_m(\Omega)$. Observe also that $u_j = \varphi_j$ near K. This implies that $\int_\Omega \chi H_m(u_j) \to \int_\Omega \chi H_m(u)$ as required.

Now, assume that $\mathcal{K} \subset \mathcal{SH}_m^-(\Omega)$ verifies (i) and (ii). Take $u \in \mathcal{K}$. We are to prove that $u \in \mathcal{E}_m(\Omega)$. Let u_j be a sequence in $\mathcal{E}_m^0(\Omega) \cap \mathcal{C}(\Omega)$ such that $u_j \downarrow u$ on Ω. This can be done thanks to the global regularization theorem. Consider a relatively compact $B \Subset \Omega$ and set for each j

$$h_j := \sup\{v \in \mathcal{SH}_m^-(\Omega) \ / \ v \leq u_j \text{ on } \ B\}.$$

Then, $h_j \in \mathcal{E}_m^0(\Omega)$ and $\operatorname{supp} H_m(h_j) \subset \bar{B}$, for every j. Furthermore, $h_j \downarrow u$ on B and $\sup_j \int_\Omega H_m(h_j) = \sup_j \int_{\bar{B}} H_m(h_j) < +\infty$ since $H_m(h_j)$ is weakly convergent in view of (ii). $\square$

Remark 3.6. By Theorem 3.5 each $u \in \mathcal{E}_m(\Omega)$ is localy in $\mathcal{F}_m(\Omega)$, i.e, for each $K \Subset \Omega$ there exists $\tilde{u} \in \mathcal{F}_m(\Omega)$ such that $\tilde{u} = u$ on K.

Definition 3.7. We define the p-energy ($p > 0$) of $\varphi \in \mathcal{E}_m^0(\Omega)$ by

$$e_p(\varphi) := \int_\Omega (-\varphi)^p H_m(\varphi).$$

We generalize Hölder inequality in the following lemma. When $m = n$ it is a result of Persson [24]. Our proof uses the same idea.

Lemma 3.8. *Let $u, v_1, ..., v_m \in \mathcal{E}_m^0(\Omega)$ and $p \geq 1$. We have*

$$\int_\Omega (-u)^p dd^c v_1 \wedge ... \wedge dd^c v_m \wedge \beta^{n-m} \leq D_{j,p} (e_p(u))^{\frac{p}{m+p}} e_p(v_1)^{\frac{1}{m+p}} ... e_p(v_m)^{\frac{1}{m+p}}, \tag{3.1}$$

where $D_{j,1} = 1$ and for each $p > 1$, $D_{j,p} := p^{p\alpha(p,m)/(p-1)}$, where

$$\alpha(p, m) = (p+2)\Big(\frac{p+1}{p}\Big)^{m-2} - p - 1.$$

Proof. Let

$$F(u, v_1, ..., v_m) = \int_\Omega (-u)^p dd^c v_1 \wedge ... \wedge dd^c v_m \wedge \beta^{n-m}, \ u, v_1, ..., v_m \in \mathcal{E}_m^0(\Omega).$$

Thanks to [24, Theorem 4.1] it suffices to prove that

$$F(u,v,v_1,...,v_{m-1}) \leq a(p) F(u,u,v_1,...,v_{m-1})^{\frac{p}{p+1}} F(v,v,v_1,...,v_{m-1})^{\frac{1}{p+1}}, \tag{3.2}$$

where $a(p)=1$ if $p=1$ and $a(p)=p^{\frac{p}{p-1}}$ if $p>1$. Set $T = dd^c v_1 \wedge ... \wedge dd^c v_{m-1} \wedge \beta^{n-m}$. When $p=1$, (3.2) becomes

$$\int_\Omega (-u) dd^c v \wedge T \leq \Big(\int_\Omega (-u) dd^c u \wedge T\Big)^{\frac{1}{2}} \Big(\int_\Omega (-v) dd^c v \wedge T\Big)^{\frac{1}{2}},$$

which is Cauchy-Schwarz inequality. In the case $p>1$, we repeat the proof of Proposition 2.5 to obtain

$$\int_\Omega (-u)^p dd^c v \wedge T \leq p \int_\Omega (-u)^{p-1}(-v) dd^c u \wedge T.$$

By Hölder inequality we get

$$\int_\Omega (-u)^p dd^c v \wedge T \leq p\Big(\int_\Omega (-u)^p dd^c u \wedge T\Big)^{\frac{p-1}{p}} \Big(\int_\Omega (-v)^p dd^c u \wedge T\Big)^{\frac{1}{p}}. \tag{3.3}$$

By interchanging u and v we obtain

$$\int_\Omega (-v)^p dd^c u \wedge T \leq p\Big(\int_\Omega (-u)^p dd^c v \wedge T\Big)^{\frac{1}{p}} \Big(\int_\Omega (-v)^p dd^c v \wedge T\Big)^{\frac{p-1}{p}}. \tag{3.4}$$

Combining (3.4) and (3.3) we obtain the result.

□

Thanks to Lemma 3.8 we can bound $\int_\Omega (u_0)^p dd^c u_1 \wedge ... \wedge dd^c u_m \wedge \beta^{n-m}$ by $e_p(u_j), j=0,...,m$ if $p \geq 1$. To get similar estimates when $p \in (0,1)$ we refer to [12].

Lemma 3.9. *Let $u,v \in \mathcal{E}^0_m(\Omega)$ and $0<p<1$. If T is a closed m-positive current of type $T = dd^c v_1 \wedge ... \wedge dd^c v_{m-k} \wedge \beta^{n-m}$, where $u_j \in \mathcal{SH}_m(\Omega) \cap L^\infty_{\text{loc}}$. Then*

$$\int_\Omega (-u)^p (dd^c v)^k \wedge T \leq 2\int_\Omega (-u)^p (dd^c u)^k \wedge T + 2\int_\Omega (-v)^p (dd^c v)^k \wedge T.$$

Proof. We follow the proof of [12, Proposition 2.5]. Set $\chi(t) = -(-t)^p : \mathbb{R}^- \to \mathbb{R}^-$ and observe that $\chi'(2t) \leq \chi'(t), \ \forall t<0$. Then

$$\begin{aligned}
\int_\Omega (-\chi)\circ u \, (dd^c v)^k \wedge T &= \int_{-\infty}^0 \chi'(t)(dd^c v)^k \wedge T(u<t)dt \\
&\leq 2\int_{-\infty}^0 \chi'(t)(dd^c v)^k \wedge T(u<2t)dt.
\end{aligned}$$

Since $(u<2t) \subset (u<v+t) \cup (v<t)$, we get

$$\int_\Omega (-\chi)\circ u \, (dd^c v)^k \wedge T \leq 2\int_{-\infty}^0 \chi'(t)(dd^c v)^k \wedge T(u<v+t)dt + 2\int_\Omega (-\chi)\circ v \, (dd^c v)^k \wedge T.$$

The comparison principle yields

$$(dd^c v)^k \wedge T(u<v+t) \leq (dd^c u)^k \wedge T(u<v+t).$$

Now, it suffices to note that $(u<v+t) \subset (u<t)$. □

Proposition 3.10. *Let $0 < p < 1$. There exists $C_p > 0$ such that*

$$0 \leq \int_\Omega (-\varphi_0)^p dd^c\varphi_1 \wedge \cdots \wedge dd^c\varphi_m \wedge \beta^{n-m} \leq C_p \max_{0\leq j\leq m} e_p(\varphi_j),$$

for all $0 \geq \varphi_0, \ldots, \varphi_m \in \mathcal{E}_m^0(\Omega)$.

Proof. By applying Lemma 3.9 with $u = \varphi_0$, $v = \varphi_1$ and $T = dd^c\varphi_2 \wedge \cdots \wedge dd^c\varphi_m \wedge \beta^{n-m}$ we obtain

$$\begin{aligned} \int_\Omega (-\varphi_0)^p dd^c\varphi_1 \wedge T &\leq 2\int_\Omega (-\varphi_0)^p dd^c\varphi_0 \wedge T \\ &+ 2\int_\Omega (-\varphi_1)^p \, dd^c\varphi_1 \wedge T. \end{aligned} \tag{3.5}$$

Next, we can assume that $\varphi_0 = \varphi_1$. Set $u = \epsilon \sum_{i=1}^m \varphi_i$, where $\epsilon > 0$ is quite small. Observe that

$$(dd^c u)^m \wedge \beta^{n-m} \geq \epsilon^m dd^c\varphi_1 \wedge \cdots \wedge dd^c\varphi_m \wedge \beta^{n-m}. \tag{3.6}$$

It suffices to get control on $\int_\Omega (-\varphi_i)^p \, H_m(u),\ 1 \leq i \leq m$ to conclude. By using again Lemma 3.9 we get

$$\int_\Omega (-\varphi_i)^p \, H_m(u) \leq 2e_p(\varphi_i) + 2e_p(u),$$

where $e_p(u) := \int_\Omega (-u)^p \, H_m(u)$.

By subadditivity and homogeneity of $t \mapsto t^p$, we have

$$e_p(u) \leq \epsilon^p \sum_{j=1}^m \int_\Omega (-\varphi_j)^p H_m(u),$$

hence

$$\sum_{i=1}^m \int_\Omega (-\varphi_i)^p \, H_m(u) \leq \frac{2}{1 - 2m\epsilon^p} \sum_{i=1}^m e_p(\varphi_i). \tag{3.7}$$

From (3.5), (3.6) and (3.7) we get

$$\int_\Omega (-\varphi_0)^p dd^c\varphi_1 \wedge \cdots \wedge dd^c\varphi_m \wedge \beta^{n-m} \leq \frac{4m}{\epsilon^m[1 - 2m\epsilon^p]} \max_{1\leq i\leq m} e_\chi(\varphi_i),$$

from which the result follows. □

From Lemma 3.8 and Proposition 3.10 we easily get the following result.

Corollary 3.11. *Let (u_j) be a sequence in $\mathcal{E}_m^0(\Omega)$ and $p > 0$. If $\sup_j e_p(u_j) < +\infty$, then*

$$u = \sum_{j=1}^\infty 2^{-j} u_j \ \text{belongs to} \ \mathcal{E}_m^p(\Omega).$$

We are now in the position to prove the convexity of Cegrell' classes.

Theorem 3.12. *By $\mathcal{E}$ we denote one of the classes $\mathcal{E}_m^0(\Omega)$, $\mathcal{E}_m(\Omega)$, $\mathcal{F}_m(\Omega)$, $\mathcal{E}_m^p(\Omega)$, $\mathcal{F}_m^p(\Omega)$, $p > 1$. They are convex and moreover, if $v \in \mathcal{E}$, $u \in \mathcal{SH}_m^-(\Omega)$, $u \geq v$, then $u \in \mathcal{E}$.*

Proof. **The class $\mathcal{E}^0_m(\Omega)$.** Let $u,v\in\mathcal{E}^0_m(\Omega)$. We claim that

$$\int_{(u=tv)} H_m(u+v)=0,$$

for almost everywhere $t>0$. In fact, the function $f(t)=\mu(u<tv), t>0$ is decreasing and right-continuous since $\mu=H_m(u+v)$ is a Borel measure. The fact that $f(t)<+\infty, \forall t>0$ follows from the comparison principle as follow:

$$\int_{u<tv} H_m(u+v)=\int_{\frac{1+t}{t}u<u+v} H_m(u+v)\le\int_{\frac{1+t}{t}u<u+v}\frac{(1+t)^m}{t^m}H_m(u)<+\infty.$$

Therefore, $\lim_{t\to t_0^-} f(t)=\mu(u\le t_0v)$ which implies that the set $I_\mu:=\{t>0\ /\ \mu(u=tv)>0\}$ coincides with the discontinuity set of f. The latter being a decreasing function, we deduce that I_μ is at most countable. This prove the claim.

Fix such a $t>0$ and apply again the comparison principle to obtain

$$\begin{aligned}\int_\Omega H_m(u+v) &= \int_{u-tv<0} H_m(u+v)+\int_{u-tv>0} H_m(u+v)\\ &= \int_{\frac{1+t}{t}u<u+v} H_m(u+v)+\int_{(1+t)v<u+v} H_m(u+v)\\ &\le \frac{(1+t)^m}{t^m}\int_{\frac{1+t}{t}u<u+v} H_m(u)+(1+t)^m\int_{(1+t)v<u+v} H_m(v)\\ &\le \frac{(1+t)^m}{t^m}\int_\Omega H_m(u)+(1+t)^m\int_\Omega H_m(v)<+\infty.\end{aligned}$$

Thus, $u+v\in\mathcal{E}^0_m(\Omega)$, from which $\mathcal{E}^0_m(\Omega)$ is convex.

Now, assume that $u\in\mathcal{E}^0_m(\Omega)$ and $v\in\mathcal{SH}^-_m(\Omega)$. Set $w=\max(u,v)$. We are to prove that $H_m(w)$ has finite mass. Fix $h\in\mathcal{E}^0_m(\Omega)$ such that $-1\le h\le 0$. Integrating by parts we get

$$\int_\Omega hH_m(w)\ge\int_\Omega hH_m(u).$$

Letting $h\downarrow -1$ we see that $\int_\Omega H_m(w)\le\int_\Omega H_m(u)$.

The same arguments can be used to prove the result for the classes $\mathcal{E}_m(\Omega),\mathcal{F}_m(\Omega)$.

The class $\mathcal{E}^p_m(\Omega)$. Assume that $u,v\in\mathcal{E}^p_m(\Omega)$ and u_j,v_j two sequences in $\mathcal{E}^0_m(\Omega)$ which decrease to u,v respectively and satisfy

$$\sup_j\max\Big(\int_\Omega(-u_j)^pH_m(u_j),\int_\Omega(-v_j)^pH_m(v_j)\Big)<+\infty.$$

We are to prove that

$$\sup_j\int_\Omega(-u_j-v_j)^pH_m(u_j+v_j)<+\infty.$$

By Hölder inequality it remains to get control on the terms

$$\int_\Omega(-u_j)^p(dd^cu_j)^k\wedge(dd^cv_j)^{m-k}\wedge\beta^{n-m}\ \text{et}\ \int_\Omega(-v_j)^p(dd^cu_j)^k\wedge(dd^cv_j)^{m-k}\wedge\beta^{n-m}.$$

The latter follows from Lemma 3.8 and Proposition 3.10.

Now, let $u \in \mathcal{SH}_m^-(\Omega)$, $v \in \mathcal{E}_m^p(\Omega)$ and suppose that $u \geq v$. Take a sequence $(v_j) \subset \mathcal{E}_m^0(\Omega)$ and a sequence $(u_j) \in \mathcal{E}_m^0(\Omega) \cap \mathcal{C}(\Omega)$ such that $v_j \downarrow v$, $u_j \downarrow u$, $u_j \geq v_j$ and

$$\sup_j e_p(v_j) < +\infty.$$

If $p \geq 1$, Lemma 3.8 gives us

$$e_p(u_j) \leq \int_\Omega (-v_j)^p H_m(u_j) \leq C.e_p(v_j)^{\frac{p}{m+p}}.e_p(u_j)^{\frac{m}{m+p}}.$$

We then deduce that $\sup_j e_p(u_j) < +\infty$, and $u \in \mathcal{E}_m^p(\Omega)$.

If $0 < p < 1$, for each j, set $h_j = -(-v_j)^p$. Then h_j is bounded, m-sh and vanishes on the boundary. Integrating by parts, we get

$$e_p(u_j) \leq \int_\Omega (-h_j) H_m(u_j) \leq \int_\Omega (-h_j) H_m(v_j) = e_p(v_j).$$

Combining the two steps above we get the results for $\mathcal{F}_m^p(\Omega)$. $\square$

3.2. **Definition of the complex Hessian operator.** In this section we prove that the complex Hessian operator $H_m(u)$ is well-defined for all $u \in \mathcal{E}_m(\Omega) \cup_{p>0} \mathcal{E}_m^p(\Omega)$. We first prove that continuous functions in $\mathcal{E}_m^0(\Omega)$ can be used as test functions.

Lemma 3.13. $\mathcal{C}_0^\infty(\Omega) \subset \mathcal{E}_m^0(\Omega) \cap \mathcal{C}(\Omega) - \mathcal{E}_m^0(\Omega) \cap \mathcal{C}(\Omega)$.

Proof. Fix $\chi \in \mathcal{C}_0^\infty(\Omega)$ and $0 > \psi \in \mathcal{E}_m^0(\Omega)$. We can choose $A > 0$ big enough such that $\chi + A|z|^2$ is plurisubharmonic. Take $a, b \in \mathbb{R}$ such that

$$a < \inf \chi < \sup_\Omega (|\chi| + A|z|^2) < b.$$

Consider

$$\varphi_1 = \max(\chi + A|z|^2 - b, B\psi); \quad \varphi_2 = \max(A|z|^2 - b, B\psi),$$

where B is big enough such that $B\psi < a - b$ in $\mathrm{supp}(\chi)$. We can easily check that $\varphi_1, \varphi_2 \in \mathcal{E}_m^0(\Omega)$ and $\chi = \varphi_1 - \varphi_2$. $\square$

Theorem 3.14. *Let $u^p \in \mathcal{E}_m(\Omega), p = 1, ..., m$ and $(g_j^p)_j \subset \mathcal{E}_m^0(\Omega)$ such that $g_j^p \downarrow u^p, \forall p$. Then the sequence of measures*

$$dd^c g_j^1 \wedge dd^c g_j^2 \wedge ... \wedge dd^c g_j^m \wedge \beta^{n-m}$$

converges weakly to a positive Radon measure which does not depend on the choice of the sequences (g_j^p). We then define $dd^c u^1 \wedge ... \wedge dd^c u^m \wedge \beta^{n-m}$ to be this weak limit.

Proof. Suppose first that $\sup_j \int_\Omega H_m(g_j^p) < +\infty$. Then for each $h \in \mathcal{E}_m^0(\Omega)$, the sequence

$$\int_\Omega h dd^c g_j^1 \wedge dd^c g_j^2 \wedge ... \wedge dd^c g_j^m \wedge \beta^{n-m}$$

is decreasing. Moreover,

$$\int_\Omega h H_m(g_j^p) \geq (\inf_\Omega h) \sup_j \int_\Omega H_m(g_j^p) > -\infty.$$

Thus we see that $\lim_j \int_\Omega h dd^c g_j^1 \wedge dd^c g_j^2 \wedge ... \wedge dd^c g_j^m \wedge \beta^{n-m}$ exists pour every $h \in \mathcal{E}_m^0(\Omega)$.

As consequence, $dd^cg_j^1\wedge dd^cg_j^2\wedge...\wedge dd^cg_j^m\wedge\beta^{n-m}$ is weakly convergent. Suppose now that $(v_j^p)_j$ are other sequences which decrease to $u^p, p=1,...,m$. We have

$$\begin{aligned}
&\int_\Omega hdd^cv_j^1\wedge dd^cv_j^2\wedge...\wedge dd^cv_j^m\wedge\beta^{n-m}\\
&=\int_\Omega v_j^1dd^ch\wedge dd^cv_j^2\wedge...\wedge dd^cv_j^m\wedge\beta^{n-m}\\
&\geq\int_\Omega u^1dd^ch\wedge dd^cv_j^2\wedge...\wedge dd^cv_j^m\wedge\beta^{n-m}\\
&=\lim_{s_1\to+\infty}\int_\Omega g_{s_1}^1dd^ch\wedge dd^cv_j^2\wedge...\wedge dd^cv_j^m\wedge\beta^{n-m}\\
&=\lim_{s_1\to+\infty}\int_\Omega v_j^2dd^ch\wedge dd^cg_{s_1}^1\wedge...\wedge dd^cv_j^m\wedge\beta^{n-m}\geq...\\
&\geq\lim_{s_1\to+\infty}\lim_{s_2\to+\infty}\cdots\lim_{s_m\to+\infty}\int_\Omega h_jdd^cg_{s_1}^1\wedge dd^cg_{s_2}^2\wedge...\wedge dd^cg_{s_m}^m\wedge\beta^{n-m}\\
&=\lim_{s\to+\infty}\int_\Omega hdd^cg_s^1\wedge dd^cg_s^2\wedge...\wedge dd^cg_m^s\wedge\beta^{n-m}.
\end{aligned}$$

From this we deduce that $\lim_j\int_\Omega hdd^cv_j^1\wedge dd^cv_j^2\wedge...\wedge dd^cv_j^m\wedge\beta^{n-m}$ exists and the limit is not less than

$$\lim_{j\to+\infty}\int_\Omega hdd^cg_j^1\wedge dd^cg_j^2\wedge...\wedge dd^cg_j^m\wedge\beta^{n-m}.$$

By interchanging the role of g_j^p and v_j^p) we obtain the equality.

It remains to remove the hypothesis $\sup_j\int_\Omega H_m(g_j^p)<+\infty$. Without loss of generality we can assume that g_j^p are continuous. Let K be a compact subset of Ω. We cover K by W_q, $q=1,...,N$ and fix $(h_j^{pq})_j, p=1,..,m; q=1,..,N$ sequences which converge to u^p in W_q as in the definition of $\mathcal{E}_m(\Omega)$. Set $w_j^p:=\sum_1^N h_j^{pq}$. We can rearrange the sequence h_j^{pq} such that $w_j^p\leq g_j^p$ on $\bigcup_q W_q$. It is clear that $w_j^p\in\mathcal{E}_m^0(\Omega)$ and $\sup_j\int_\Omega H_m(w_j^p)<+\infty$. By setting $v_j^p=\max(g_j^p,w_j^p)$, we get $\sup_j\int_\Omega H_m(v_j^p)<+\infty$ and $v_j^p=g_j^p$ near K. This completes the proof. □

Corollary 3.15. *Let $u_1,...,u_m\in\mathcal{F}_m(\Omega)$ and $u_1^j,...,u_m^j$ sequences of functions in $\mathcal{E}_m^0(\Omega)\cap\mathcal{C}(\Omega)$ decreasing to $u_1,...,u_m$ respectively such that*

$$\sup_{j,p}\int_\Omega H_m(u_j^p)<+\infty.$$

Then for each $\varphi\in\mathcal{E}_m^0(\Omega)\cap\mathcal{C}(\Omega)$ we have

$$\lim_{j\to+\infty}\int_\Omega\varphi dd^cu_1^j\wedge...\wedge dd^cu_m^j\wedge\beta^{n-m}=\int_\Omega\varphi dd^cu_1\wedge...\wedge dd^cu_m\wedge\beta^{n-m}.$$

Proof. Il is clear that

$$\sup_j\int_\Omega dd^cu_1^j\wedge...\wedge dd^cu_m^j\wedge\beta^{n-m}<+\infty. \tag{3.8}$$

Fix $\epsilon>0$ small enough and consider $\varphi_\epsilon=\max(\varphi,-\epsilon)$. The function $\varphi-\varphi_\epsilon$ is continuous with compact support in Ω. It follows from Theorem 3.14 that

$$\lim_{j\to+\infty}\int_\Omega(\varphi-\varphi_\epsilon)dd^cu_1^j\wedge...\wedge dd^cu_m^j\wedge\beta^{n-m}=\int_\Omega(\varphi-\varphi_\epsilon)dd^cu_1\wedge...\wedge dd^cu_m\wedge\beta^{n-m}.$$

Observe also that $|\varphi_\epsilon| \le \epsilon$. By using (3.8), we get the result. $\square$

Corollary 3.16. *Assume that $(u_j) \subset \mathcal{E}_m^0(\Omega)$ decreases to u such that*

$$\sup_j \int_\Omega H_m(u_j) < +\infty$$

Then for every $h \in \mathcal{E}_m^0(\Omega)$ we have

$$hH_m(u_j) \rightharpoonup hH_m(u).$$

Proof. For every test function χ the function $h\chi$ is upper semicontinuous. Thus,

$$\liminf_{j\to+\infty} \int_\Omega (-h)\chi H_m(u_j) \ge \int_\Omega (-h)\chi H_m(u).$$

Let Θ is any cluster point of this the sequence $(-h)H_m(u_j)$. From the inequality above we deduce that $\Theta \ge (-h)H_m(u)$. Moreover, it follows from Corollary 3.15 that the sequence $\int_\Omega (-h)H_m(u_j)$ increases to $\int_\Omega (-h)H_m(u)$. This implies that the total mass of Θ is less than or equal to the total mass of $(-h)H_m(u)$ and hence these measures are equal. $\square$

The same arguments can be used for the classes $\mathcal{E}_m^p(\Omega), p > 0$.

Theorem 3.17. *Let $u^1, ..., u^m \in \mathcal{E}_m^p(\Omega),\ p > 0$ and $(g_j^i)_j \subset \mathcal{E}_m^0(\Omega)$ be such that $g_j^i \downarrow u^i, \forall i = 1, ..., m$ and*

$$\sup_{i,j} e_p(g_j^i) < +\infty.$$

Then the sequence of measures $dd^c g_j^1 \wedge dd^c g_j^2 \wedge ... \wedge dd^c g_j^m \wedge \beta^{n-m}$ converges weakly to a positive Radon measure which does not depend on the choice of the sequences (g_j^i). We then define $dd^c u^1 \wedge ... \wedge dd^c u^m \wedge \beta^{n-m}$ to be this weak limit.

Proof. Fix $h \in \mathcal{E}_m^0(\Omega)$. Then

$$\int_\Omega h dd^c g_j^1 \wedge dd^c g_j^2 \wedge ... \wedge dd^c g_j^m \wedge \beta^{n-m}$$

is decreasing. From Lemma 3.8 and Proposition 3.10 we get

$$\sup_j \int_\Omega (-h) dd^c g_j^1 \wedge ... \wedge dd^c g_j^m \wedge \beta^{n-m} < +\infty.$$

Thus the limit $\lim_j \int_\Omega h dd^c g_j^1 \wedge dd^c g_j^2 \wedge ... \wedge dd^c g_j^m \wedge \beta^{n-m}$ exists for every $h \in \mathcal{E}_m^0(\Omega)$.

This implies the weak convergence of the sequence

$$dd^c g_j^1 \wedge dd^c g_j^2 \wedge ... \wedge dd^c g_j^m \wedge \beta^{n-m}$$

in view of Lemma 3.13. To prove the remaining it suffices to repeat the proof of Theorem 3.14. $\square$

Corollary 3.18. *Let $u_1, ..., u_m \in \mathcal{E}_m^p(\Omega), p > 0$ and $u_1^j, ..., u_m^j$ sequences of functions in $\mathcal{E}_m^0(\Omega)$ decreasing to $u_1, ..., u_m$ respectively such that*

$$\sup_{j,k} \int_\Omega (-u_k^j)^p H_m(u_k^j) < +\infty.$$

Then for each $\varphi \in \mathcal{E}_m^0(\Omega) \cap \mathcal{C}(\Omega)$ we have

$$\lim_{j\to+\infty} \int_\Omega \varphi dd^c u_1^j \wedge ... \wedge dd^c u_m^j \wedge \beta^{n-m} = \int_\Omega \varphi dd^c u_1 \wedge ... \wedge dd^c u_m \wedge \beta^{n-m}.$$

Proof. As in the proof of Corollary 3.15, it suffices to prove that

$$\lim_{j\to+\infty}\int_\Omega(-\varphi_\epsilon)dd^cu_1^j\wedge...\wedge dd^cu_m^j\wedge\beta^{n-m}=0,$$

where $\varphi_\epsilon=\max(\varphi,-\epsilon)$. The left-hand side term is dominated by

$$\epsilon^{1-p}\int_\Omega(-\varphi_\epsilon)^p dd^cu_1^j\wedge...\wedge dd^cu_m^j\wedge\beta^{n-m}.$$

We know that $\varphi_\epsilon=\varphi$ near the boundary of Ω. Thus,

$$e_p(\varphi_\epsilon)=\int_\Omega(-\varphi_\epsilon)^pH_m(\varphi_\epsilon)\leq\epsilon^p\int_\Omega H_m(\varphi).$$

By applying Lemma 3.8 and Proposition 3.10, we get the result. $\square$

3.3. **Integration by parts.** From Theorem 3.14 and Corollary 3.15 we prove the integration by parts formula for functions in $\mathcal{E}_m^p(\Omega)$, $p>0$ and $\mathcal{F}_m(\Omega)$.

Theorem 3.19.

$$\int_\Omega udd^cv\wedge T=\int_\Omega vdd^cu\wedge T,$$

where $u,v,\varphi_1,...,\varphi_p\in\mathcal{F}_m(\Omega)$ *and* $T=dd^c\varphi_1\wedge...\wedge dd^c\varphi_p\wedge\beta^{n-m-1}$.

Proof. Let $u_j,v_j,\varphi_1^j,..,\varphi_{m-1}^j$ be sequences in $\mathcal{E}_m^0(\Omega)\cap\mathcal{C}(\Omega)$ decreasing to $u,v,\varphi_1,...,\varphi_{m-1}$ respectively such that their total masses are uniformly bounded:

$$\sup_j\int_\Omega dd^cv_j\wedge T_j<+\infty,\ \ \sup_j\int_\Omega dd^cu_j\wedge T_j<+\infty,$$

where $T_j=dd^c\varphi_1^j\wedge...\wedge dd^c\varphi_{m-1}^j\wedge\beta^{n-m}$. Theorem 3.14 gives us that $dd^cu_j\wedge T_j\rightharpoonup dd^cu\wedge T$. For each fixed $k\in\mathbb{N}$ and any $j>k$ we have

$$\int_\Omega v_kdd^cu_k\wedge T_k\geq\int_\Omega v_kdd^cu_j\wedge T_j\geq\int_\Omega v_jdd^cu_j\wedge T_j.$$

We then deduce that the sequence of real numbers $\int_\Omega v_jdd^cu_j\wedge T$ decreases to some $a\in\mathbb{R}\cup\{-\infty\}$. By letting $j\to+\infty$ we get

$$\int_\Omega v_kdd^cu\wedge T\geq a,$$

from which we obtain $\int_\Omega vdd^cu\wedge T\geq a$. For each fixed k we also have

$$\begin{aligned}\int_\Omega vdd^cu\wedge T &\leq \int_\Omega v_kdd^cu\wedge T=\lim_{j\to+\infty}\int_\Omega v_kdd^cu_j\wedge T_j\\ &\leq \int_\Omega v_kdd^cu_k\wedge T_k.\end{aligned}$$

This implies that $\int_\Omega vdd^cu\wedge T=a$, from which the result follows. $\square$

We obtain the same result for the classes $\mathcal{E}_m^p(\Omega),p>0$ by using the same arguments.

Theorem 3.20. *Integration by parts is allowed in* $\mathcal{E}_p,p>0$. *Precisely, assume that* $u,v\in\mathcal{E}_m^p(\Omega)$ *and* T *is a closed m-positive current of type* $T=dd^c\varphi_1\wedge...dd^c\varphi_{m-1}\wedge\beta^{n-m}$, *where* $\varphi_j\in\mathcal{E}_m^p(\Omega),\forall j$. *Then*

$$\int_\Omega udd^cv\wedge T=\int_\Omega vdd^cu\wedge T.$$

3.4. **Comparison principle.** In this section we prove that the comparison principle is valid in $\mathcal{E}^p_m(\Omega), p>0$.

Lemma 3.21. *Let E be an open subset of Ω and $\varphi \in \mathcal{E}^0_m(\Omega),\ p\geq 1$. Then*

$$\int_E H_m(\varphi) \leq Cap_m(E)^{\frac{p}{p+m}} e_p(\varphi)^{\frac{m}{p+m}}.$$

Proof. We can suppose that E is relatively compact in Ω. Denote by $u=u_{m,E,\Omega}$ the m-extremal function of E in Ω. Then $u\in\mathcal{E}^0_m(\Omega)$ and $u=-1$ in E. From Lemma 3.8 we have

$$\begin{aligned}\int_E H_m(\varphi) &\leq \int_\Omega (-u)^p H_m(\varphi) \leq e_p(u)^{\frac{p}{m+p}} e_p(\varphi)^{\frac{m}{m+p}}\\ &\leq \Big(\int_\Omega H_m(u)\Big)^{\frac{p}{m+p}} e_p(\varphi)^{\frac{m}{p+m}} = \mathrm{Cap}_m(E)^{\frac{p}{p+m}} e_p(\varphi)^{\frac{m}{p+m}}.\end{aligned}$$

□

Lemma 3.22. *Let $E\subset\Omega$ be an open subset and $\varphi\in\mathcal{E}^0_m(\Omega),\ 0<p<1$. Then for each $\epsilon>0$ small enough we have*

$$\int_E H_m(\varphi) \leq 2(Cap_m(E))^{1-m\epsilon} + 2Cap_m(E)^{p\epsilon} e_p(\varphi).$$

Proof. Without loss of generality we can assume that $E\Subset\Omega$. Let u be the m-extremal function of E with respect to Ω. Put $a=\mathrm{Cap}_m(E)=\int_\Omega H_m(u)$. If $a=0$, we are done. Thus, we can assume that $a>0$. By applying Lemma 3.9 we obtain

$$\begin{aligned}\int_E H_m(\varphi) &\leq a^{p\epsilon}\int_\Omega (-u/a^\epsilon)^p H_m(\varphi)\\ &\leq 2a^{p\epsilon} e_p(u/a^\epsilon) + 2a^{p\epsilon} e_p(\varphi)\\ &\leq 2a^{1-m\epsilon} + 2a^{p\epsilon} e_p(\varphi).\end{aligned}$$

□

Theorem 3.23. *Let $u,v\in\mathcal{E}^p_m(\Omega),\ p>0$ and set $A:=\{u>v\}$. Then*

$$1\!\!1_A H_m(u) = 1\!\!1_A H_m(\max(u,v)).$$

Proof. Let (u_j) be a sequence in $\mathcal{E}^0_m(\Omega)$ decreasing to u as in the definition of $\mathcal{E}^p_m(\Omega)$. From Theorem 2.17 we get

$$1\!\!1_{A_j} H_m(u_j) = 1\!\!1_{A_j} H_m(\max(u_j,v)), \tag{3.9}$$

where $A_j:=\{u_j>v\}$. Consider $\psi_j:=\max(u_j-v,0)$. Then $\psi_j\downarrow\psi:=\max(u-v,0)$, all of them are quasi-continuous.

Fix $\delta>0$ and set $g_j:=\frac{\psi_j}{\psi_j+\delta},\ g=\frac{\psi}{\psi+\delta}$. By multiplying (3.9) with g_j we obtain

$$g_j H_m(u_j) = g_j H_m(\max(u_j,v)). \tag{3.10}$$

Now, let $\chi\in\mathcal{C}^\infty_0(\Omega)$ be a test function and fix $\epsilon>0$. There exists an open subset $U\subset\Omega$ such that $\mathrm{Cap}_m(U)<\epsilon$, and there exist φ_j,φ continuous functions in Ω which coincide with ψ_j,ψ respectively on $K:=\Omega\setminus U$. The monotone convergence $\psi_j\downarrow\psi$ implies that φ_j converges uniformly to φ on $K\cap\mathrm{Supp}\chi$, which in turn implies the uniform convergence of $h_j=\frac{\varphi_j}{\varphi_j+\delta}$ on $K\cap\mathrm{Supp}\chi$ to $h=\frac{\varphi}{\varphi+\delta}$.

In the next arguments, we let C denote a positive constant which does not depend on j, ϵ. Since g_j, h_j are uniformly bounded , Lemma 3.21 and Lemma 3.22 gives us

$$\Big| \int_\Omega \chi g_j H_m(u_j) - \int_\Omega \chi h_j H_m(u_j) \Big| \leq C. \int_U H_m(u_j) \leq C.\epsilon. \tag{3.11}$$

We also obtain

$$\Big| \int_\Omega \chi g \, H_m(u) - \int_\Omega \chi h \, H_m(u) \Big| \leq C. \int_U H_m(u) \leq C. \liminf_{j \to +\infty} \int_U H_m(u_j) \leq C.\epsilon. \tag{3.12}$$

Moreover, since h is continuous on Ω and $H_m(u_j) \rightharpoonup H_m(u)$, we get

$$\lim_{j \to +\infty} \int_\Omega \chi.h(H_m(u_j) - H_m(u)) = 0.$$

Hence, we obtain

$$\limsup_{j \to +\infty} \Big| \int_\Omega \chi h_j \, H_m(u_j) - \int_\Omega \chi h \, H_m(u) \Big| \quad \leq \quad \limsup_{j \to +\infty} \int_\Omega \chi.|h_j - h| H_m(u_j).$$

Since h_j converges uniformly to h on $K \cap \mathrm{supp}\chi$, we have

$$\begin{aligned} \int_\Omega \chi.|h_j - h| H_m(u_j) \quad &= \quad \int_U \chi.|h_j - h| H_m(u_j) + \int_K \chi.|h_j - h| H_m(u_j) \\ &\leq \quad C. \int_U H_m(u_j) + \|h_j - h\|_{L^\infty(K \cap \mathrm{supp}\chi)} \int_\Omega \chi H_m(u_j). \end{aligned}$$

From the two inequalities above we get

$$\limsup_{j \to +\infty} \Big| \int_\Omega \chi h_j \, H_m(u_j) - \int_\Omega \chi h \, H_m(u) \Big| \quad \leq \quad C.\epsilon. \tag{3.13}$$

From (3.11), (3.12) and (3.13), we see that

$$\limsup_j \Big| \int_\Omega \chi g_j H_m(u_j) - \int_\Omega \chi g H_m(u) \Big| \leq C.\epsilon.$$

We have shown that $g_j H_m(u_j) \rightharpoonup g H_m(u)$. In the same way, we get

$$g_j H_m(\max(u_j, v)) \rightharpoonup g H_m(\max(u, v)),$$

and hence $g H_m(u) = g H_m(\max(u, v))$. The result follows by letting $\delta \to 0$. $\square$

Theorem 3.24. *Let $u, v \in \mathcal{E}_m^p(\Omega), p > 0$ such that $u \leq v$ on Ω. Then*

$$\int_\Omega H_m(u) \geq \int_\Omega H_m(v).$$

Proof. Let $(u_j), (v_j)$ be two sequences in $\mathcal{E}_m^0(\Omega)$ decreasing to u, v respectively such that $h \in \mathcal{E}_m^0(\Omega) \cap \mathcal{C}(\Omega)$. We can suppose that $u_j \leq v_j, \ \forall j$. Integrating by parts we get

$$\int_\Omega (-h) H_m(v_j) \leq \int_\Omega (-h) H_m(u_j).$$

Corollary 3.18 then yields

$$\lim_{j \to \infty} \int_\Omega (-h) H_m(v_j) = \int_\Omega (-h) H_m(v), \quad \text{and} \quad \lim_{j \to \infty} \int_\Omega (-h) H_m(u_j) = \int_\Omega (-h) H_m(u).$$

In consequence, we obtain

$$\int_\Omega (-h)H_m(v) \leq \int_\Omega (-h)H_m(u).$$

The result follows by letting h decrease to -1. $\square$

Theorem 3.25. *If $u \in \mathcal{E}_m^p(\Omega)$ then $e_p(u) := \int_\Omega (-u)^p H_m(u) < +\infty$. If $(u_j^i)_j, i = 0, ..., m \subset \mathcal{E}_m^0(\Omega)$, $u_j^i \downarrow u^i \in \mathcal{E}_m^p(\Omega)$ then*

$$\int_\Omega (-u_j^0) dd^c u_j^1 \wedge ... \wedge dd^c u_j^m \wedge \beta^{n-m} \nearrow \int_\Omega (-u) dd^c u^1 \wedge ... \wedge dd^c u^m \wedge \beta^{n-m}.$$

Proof. It follows from Theorem 3.17 that

$$T_j := dd^c u_j^1 \wedge ... \wedge dd^c u_j^m \wedge \beta^{n-m} \rightharpoonup T := dd^c u^1 \wedge ... \wedge dd^c u^m \wedge \beta^{n-m}.$$

Furthermore since $(-u_j^0) \uparrow (-u^0)$ all of them are lower semicontinuous, we have

$$\liminf_j \int_\Omega (-u_j^0) T_j \geq \int_\Omega (-u^0) T.$$

Thus, it suffices to prove that

$$\int_\Omega (-u^0) T_j \leq \int_\Omega (-u^0) T, \forall j.$$

Let $h \in \mathcal{E}_m^0(\Omega) \cap \mathcal{C}(\Omega)$ such that $u^0 \leq h$. By integrating by parts and using Corollary 3.18 the sequence $(\int_\Omega (-h) T_j)_j$ increases to $\int_\Omega (-h) T$ from which the result follows. $\square$

Theorem 3.26. *If $p > 0$ and $u, v \in \mathcal{E}_m^p(\Omega)$ then*

$$\int_{\{u>v\}} H_m(u) \leq \int_{\{u>v\}} H_m(v).$$

Proof. Fix $h \in \mathcal{E}_0 \cap \mathcal{C}(\Omega)$. The measure $H_m(v)$ does not charge m-polar sets. As in the proof of Proposition 3.12 we can easily show that for almost every r,

$$\int_{v=ru} (-h) H_m(v) = 0$$

This allows us to restrict ourself to the case $\int_{u=v} (-h) H_m(v) = 0$. From Theorem 3.23, we get

$$1\!\!1_{\{u>v\}} H_m(u) = 1\!\!1_{\{u>v\}} H_m(\max(u, v)), \text{ and } 1\!\!1_{\{u<v\}} H_m(v) = 1\!\!1_{\{u<v\}} H_m(\max(u, v)).$$

Furthermore, as in the proof of Theorem 3.24, we can prove that

$$\int_\Omega (-h) H_m(\max(u, v)) \leq \int_\Omega (-h) H_m(u).$$

From this we get

$$\begin{aligned} \int_{\{u>v\}} (-h) H_m(u) &= \int_{\{u>v\}} (-h) H_m(\max(u, v)) \\ &\leq \int_\Omega (-h) H_m(\max(u, v)) + \int_{\{u<v\}} h H_m(\max(u, v)) \\ &\leq \int_\Omega (-h) H_m(v) + \int_{\{u<v\}} h H_m(v) = \int_{\{u>v\}} (-h) H_m(v). \end{aligned}$$

Letting $h \downarrow -1$ we obtain the result. $\square$

Remark 3.27. We have proven above that

$$\int_{\{u>v\}}(-h)H_m(v) \le \int_{\{u>v\}}(-h)H_m(u)$$

if $u, v \in \mathcal{E}_m^p(\Omega), p > 0$ and $h \in \mathcal{E}_m^0(\Omega) \cap \mathcal{C}(\Omega)$. Thanks to the regularisation theorem (Theorem 3.1) it also holds for every $h \in \mathcal{SH}_m^-(\Omega)$.

Theorem 3.28. *Let $u, v \in \mathcal{E}_m^p(\Omega)$ $(p > 0)$ such that $H_m(u) \ge H_m(v)$. Then $u \le v$ in Ω.*

Proof. By contradiction assume that there exists $z_0 \in \Omega$ such that $v(z_0) < u(z_0)$. Let h be an exhaustion function of Ω and choose $R > 0$ such that $|z - z_0| \le R, \forall z \in \Omega$. Fix $\epsilon > 0$ small enough such that $h(z_0) < -\epsilon R^2$. The exhaustion function

$$P(z) := \max\{h(z), \epsilon(|z - z_0|^2 - R^2)\}.$$

is continuous in Ω and verifies $H_m(P) \ge \epsilon^m \beta^n$, near z_0. Take $\eta > 0$ small enough such that $v(z_0) < u(z_0) + \eta P(z_0)$. The Lebesgue measure of the set $T := \{z \in \Omega \ / \ v(z) < u(z) + \eta P(z)\} \cap B(z_0, \delta)$ is positive for every $\delta > 0$. This implies that

$$\int_T H_m(P) > 0.$$

Thus, Theorem 3.26 gives us

$$\int_T H_m(u + \eta P) \le \int_T H_m(v).$$

Moreover,

$$\int_T H_m(u + \eta P) \ge \int_T H_m(u) + \eta^m \int_T H_m(P),$$

and $H_m(v)$ is, by hypothesis, less than $H_m(u)$. We thus get a contradiction:

$$\int_T H_m(P) = 0.$$

□

4. The variational approach

In this section we use a variational method to solve the equation

$$H_m(u) = \mu,$$

where μ is a positive Radon measure. We characterize the range of $H_m(u)$ when u runs in $\mathcal{E}_m^p(\Omega)$.

The idea of this approach is to minimize the energy functional on a compact subsets of m-subharmonic functions. We then show that this minimum point is the wanted solution. Our results are direct generalizations of the classical case of plurisubharmonic functions (see [1], [5, 6]). Note also that the variational approach for the complex Monge-Ampère equation was first introduced in [4].

4.1. **The energy functional.** We recall some useful results obtained from previous sections.

- If $0 \geq u_j \downarrow u$ and $u \in \mathcal{E}_m^1(\Omega)$, then by Theorem 3.25, we have $e_1(u_j) \uparrow e_1(u)$.
- If $u, v \in \mathcal{E}_m^1(\Omega)$ and $u \leq v$ then $e_1(u) \geq e_1(v)$.

Lemma 4.1. *(i) If $(u_j) \subset \mathcal{E}_m^1(\Omega)$ then $(\sup_j u_j)^* \in \mathcal{E}_m^1(\Omega)$.*

(ii) If $(u_j) \in \mathcal{E}_m^1(\Omega)$ such that $\sup_j e_1(u_j) < +\infty$ and $u_j \downarrow u$, then $u \in \mathcal{E}_m^1(\Omega)$.

(iii) For each $C > 0$, the set $\mathcal{E}_m^{1,C} := \{u \in \mathcal{E}_m^1(\Omega) \ / \ e_1(u) \leq C\}$ is a convex compact subset of $\mathcal{SH}_m(\Omega)$.

Proof. (i) Let (φ_j) be a sequence of continuous functions in $\mathcal{E}_m^0(\Omega)$ decreasing to $\varphi := (\sup_j u_j)^*$. Since $u_j \leq \varphi_j$, we have $\sup_j e_1(\varphi_j) < +\infty$, which implies that $\varphi \in \mathcal{E}_m^1(\Omega)$.

(ii) Let (φ_j) be a sequence in $\mathcal{E}_m^0(\Omega) \cap \mathcal{C}(\Omega)$ decreasing to u. Set $\psi_j := \max(u_j, \varphi_j)$. Then $\psi_j \in \mathcal{E}_m^1(\Omega), \forall j$ and $e_1(\psi_j) \leq e_1(u_j)$. Thus, $u \in \mathcal{E}_m^1(\Omega)$.

(iii) Let (u_j) be a sequence in $\mathcal{E}_m^{1,C}$. Since $\sup_j e_1(u_j) < +\infty$, (u_j) can not tend uniformly to $-\infty$ in Ω. Thus, there exists a subsequence (still denoted by (u_j)) converging to $u \in \mathcal{SH}_m(\Omega)$ in $L^1_{\text{loc}}(\Omega)$. Set

$$\varphi_j := (\sup_{k \geq j} u_k)^* \in \mathcal{E}_m^1(\Omega), \forall j.$$

Then $\varphi_j \downarrow u$ and $\sup_j e_1(\varphi_j) \leq C$. In view of (ii), we have $u \in \mathcal{E}_m^1(\Omega)$, and since $(-\varphi_j) \uparrow (-u)$, all of them being lower semicontinuous we get

$$\int_\Omega (-u) H_m(u) \leq \liminf_j \int_\Omega (-\varphi_j) H_m(\varphi_j) \leq C.$$

This means $u \in \mathcal{E}_m^{1,C}$. □

Lemma 4.2. *Let μ be a positive Radon measure in Ω such that $\mu(\Omega) < +\infty$ and μ does not charge m-polar sets. Let (u_j) be a sequence in $\mathcal{SH}_m^-(\Omega)$ which converges in L^1_{loc} to $u \in \mathcal{SH}_m^-(\Omega)$. If $\sup_j \int_\Omega (-u_j)^2 d\mu < +\infty$ then $\int_\Omega u_j d\mu \to \int_\Omega u d\mu$.*

Proof. Since $\int_\Omega u_j d\mu$ is bounded it suffices to prove that every cluster point is $\int_\Omega u d\mu$. Without loss of generality we can assume that $\int_\Omega u_j d\mu$ converges. The sequence u_j being bounded in $L^2(\mu)$, the Banach-Saks allows us to extract a subsequence (still denoted by u_j) such that

$$\varphi_N := \frac{1}{N} \sum_{j=1}^N u_j$$

converges in $L^2(\mu)$ and μ-almost everywhere to φ. Observe also that $\varphi_N \to u$ in L^1_{loc}. For each $j \in \mathbb{N}$ set

$$\psi_j := (\sup_{k \geq j} \varphi_k)^*.$$

Then$\psi_j \downarrow u$ in Ω. But μ does not charge the m-polar set

$$\{(\sup_{k \geq j} \varphi_k)^* > \sup_{k \geq j} \varphi_k\}.$$

We thus get $\psi_j = \sup_{k \geq j} \varphi_k$ μ-almost everywhere. Therefore, ψ_j converges to φ μ-almost everywhere hence $u = \varphi$ μ-almost everywhere. This yields

$$\lim_j \int_\Omega u_j d\mu = \lim_j \int_\Omega \varphi_j d\mu = \int_\Omega u d\mu.$$

□

Lemma 4.3. *The functional $e_1 : \mathcal{E}_m^1(\Omega) \to \mathbb{R}$ is lower semicontinuous.*

Proof. Suppose that $u, u_j \in \mathcal{E}_m^1(\Omega)$ and u_j converges to u in $L^1_{\text{loc}}(\Omega)$. We are to prove that $\liminf_j e_1(u_j) \geq e_1(u)$. For each $j \in \mathbb{N}$, the function

$$\varphi_j := (\sup_{k \geq j} u_k)^*$$

belongs to $\mathcal{E}_m^1(\Omega)$ and $\varphi_j \downarrow u$. Hence $e_1(\varphi_j) \uparrow e_1(u)$. Moreover, $e_1(u_j) \geq e_1(\varphi_j)$ from which the result follows. □

Definition 4.4. A positive Radon measure μ belongs to $\mathcal{M}_1$ if there exists $A > 0$ such that

$$\int_\Omega (-\varphi) d\mu \leq A e_1(\varphi)^{1/(m+1)}, \quad \forall \varphi \in \mathcal{E}_m^1(\Omega).$$

The functional $\mathcal{F}_\mu : \mathcal{E}_m^1(\Omega) \to \mathbb{R}$ is defined by

$$\mathcal{F}_\mu(u) = \frac{1}{m+1} \int_\Omega (-u) H_m(u) + \int_\Omega u d\mu.$$

Lemma 4.5. *If $u, v \in \mathcal{E}_m^1(\Omega)$ then*

$$e_1(u+v)^{\frac{1}{m+1}} \leq e_1(u)^{\frac{1}{m+1}} + e_1(v)^{\frac{1}{m+1}}.$$

Furthermore, $\mathcal{F}_\mu$ is convex if $\mu \in \mathcal{M}_1$. In this case, $\mathcal{F}_\mu$ is proper, i.e $\mathcal{F}_\mu(u_j) \to +\infty$ if $e_1(u_j) \to +\infty$.

Proof. It follows from Lemma 3.8 that

$$e_1(u+v) \leq e_1(u)^{\frac{1}{m+1}} e_1(u+v)^{\frac{m}{m+1}} + e_1(v)^{\frac{1}{m+1}} e_1(u+v)^{\frac{m}{m+1}}$$

which implies that $e_1^{\frac{1}{m+1}}$ is convex, so is e_1. From the definition of $\mathcal{M}_1$, there exists $A > 0$ such that

$$\|u\|_{L^1(\mu)} \leq A e_1(u)^{\frac{1}{1+m}}, \qquad \text{for every } u \in \mathcal{E}_m^1(\Omega).$$

We thus obtain

$$\mathcal{F}_\mu(u_j) = \frac{1}{m+1} e_1(u_j) - \|u_j\|_{L^1(\mu)} \geq \frac{1}{n+1} e_1(u_j) - A e_1(u_j)^{\frac{1}{m+1}} \to \infty.$$

□

Let $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ be an upper semicontinuous function. Suppose that there exists $w \in \mathcal{E}_m^1(\Omega)$ such that $w \leq u$. we define the projection of u on $\mathcal{E}_m^1(\Omega)$ by

$$P(u) := \sup\{v \in \mathcal{E}_m^1(\Omega) \ / \ v \leq u\}.$$

Lemma 4.6. *Let $u, v \in \mathcal{E}_m^1(\Omega)$ and suppose that v is continuous. For each $t < 0$, the function $P(u+tv)$ belongs to $\mathcal{E}_m^1(\Omega)$, and for each $s < 0$,*

$$|P(u+tv) - P(u+sv)| \leq |t-s|(-v).$$

Proof. Fix $t < 0$. The function $P(u+tv)$ is upper semicontinuous. It is clear that $u \leq P(u+tv) \leq u + tv$, which implies that $P(u+tv) \in \mathcal{E}_m^1(\Omega)$. For every $s < t$ we have

$$P(u+tv) \leq P(u+sv), \quad \text{and} \quad P(u+sv) + (t-s)v \leq P(u+tv).$$

Hence, $|P(u+tv) - P(u+sv)| \leq |t-s|(-v)$. □

Lemma 4.7. *Let $u : \Omega \to \mathbb{R}$ be a continuous function. Suppose that there exists $w \in \mathcal{E}^1_m(\Omega)$ such that $w \le u$. Then*

$$\int_{\{P(u)<u\}} H_m(P(u)) = 0. \tag{4.1}$$

Proof. Without loss of generality we can assume that w is bounded. From Choquet's lemma, there exists an increasing sequence $(u_j) \subset \mathcal{E}^1_m(\Omega) \cap L^\infty(\Omega)$ such that

$$(\lim_j u_j)^* = P(u).$$

Let $x_0 \in \{P(u) < u\}$. Since u is continuous, there exists $\epsilon > 0, r > 0$ such that

$$P(u)(x) < u(x_0) - \epsilon < u(x), \quad \forall x \in B = B(x_0, r).$$

For each fixed j, by approximating $u_j|_{\partial B}$ from above by a sequence of continuous functions on ∂B and by using [7, Theorem 2.10], we can find a function $\varphi_j \in \mathcal{SH}_m(B)$ such that $\varphi_j = u_j$ on ∂B and $H_m(\varphi_j) = 0$ in B. The comparison principle gives us $\varphi_j \ge u_j$ in B. The function ψ_j, defined by $\psi_j = \varphi_j$ in B and $\psi_j = u_j$ in $\Omega \setminus B$, belongs to $\mathcal{E}^1_m(\Omega) \cap L^\infty(\Omega)$. For each $x \in \partial B$ we have $\varphi_j(x) = u_j(x) \le P(u)(x) \le u(x_0) - \epsilon$. We thus deduce that $\varphi_j \le u(x_0) - \epsilon$ in B since $u(x_0) - \epsilon$ is a constant and φ_j is m-sh. Hence, $u_j \le \psi_j \le u$ inΩ. This implies

$$(\lim \psi_j)^* = P(u).$$

It follows from Theorem 2.15 that $H_m(\psi_j) \rightharpoonup H_m(P(u))$. Therefore,

$$H_m(P(u))(B) \le \liminf_{j\to+\infty} H_m(\psi_j)(B) = 0,$$

from which the result follows. ∎

Lemma 4.8. *Let $u, v \in \mathcal{E}^1_m(\Omega)$ and suppose that v is continuous. For each $t < 0$, we define*

$$h_t = \frac{P(u + tv) - tv - u}{t}.$$

Then for each $0 \le k \le m$,

$$\lim_{t\nearrow 0} \int_\Omega h_t (dd^c u)^k \wedge (dd^c P(u + tv))^{m-k} \wedge \beta^{n-m} = 0\,. \tag{4.2}$$

In particular,

$$\lim_{t\nearrow 0} \int_\Omega \frac{P(u + tv) - u}{t} (dd^c u)^k \wedge (dd^c P(u + tv))^{m-k} \wedge \beta^{n-m} = \int_\Omega v H_m(u)\,. \tag{4.3}$$

Proof. An easy computation shows that h_t is decreasing in t and $0 \le h_t \le -v$. For each fixed $s < 0$ we have

$$\begin{aligned}
&\lim_{t\nearrow 0} \int_\Omega h_t (dd^c u)^k \wedge (dd^c P(u + tv))^{m-k} \wedge \beta^{n-m} \\
&\le \lim_{t\nearrow 0} \int_\Omega h_s (dd^c u)^k \wedge (dd^c P(u + tv))^{m-k} \wedge \beta^{n-m} \\
&= \int_\Omega h_s (dd^c u)^m \wedge \beta^{n-m} \le \int_{\{P(u+sv)-sv<u\}} (-v)(dd^c u)^m \wedge \beta^{n-m}.
\end{aligned}$$

Let $u_k \in \mathcal{E}_m^0(\Omega) \cap C(\Omega)$ be a sequence decreasing to u such that

$$\int\limits_{\{P(u+sv)-sv<u\}} (-v)(dd^c u)^m \wedge \beta^{n-m} \leq 2 \int\limits_{\{P(u_k+sv)-sv<u\}} (-v)(dd^c u)^m \wedge \beta^{n-m} .$$

Taking into account Remark 3.27 and Lemma 4.7 we can conclude that

$$\begin{aligned} &\int\limits_{\{P(u_k+sv)-sv<u\}} (-v)(dd^c u)^m \wedge \beta^{n-m} \\ &\leq \int\limits_{\{P(u_k+sv)-sv<u_k\}} (-v)(dd^c(P(u_k+sv)-sv))^m \wedge \beta^{n-m} \\ &\leq -sM \to 0\,, \qquad \text{as } s \to 0. \end{aligned}$$

Here, M stands for a positive constant which depends only on m, $\|v\|$, and $\int_\Omega v(dd^c(u+v))^m \wedge \beta^{n-m}$. Equality (4.3) follows from equality (4.2). The proof is thus complete. □

Lemma 4.9. *Let $u, v \in \mathcal{E}_m^1(\Omega)$, and assume that v is continuous. The function*

$$g(t) = e_1(P(u+tv)),\ t \in \mathbb{R}$$

is differentiable at 0 and

$$g'(0) = (m+1)\int_\Omega (-v) H_m(u).$$

Proof. If $t > 0$, $P(u+tv) = u+tv$. It is easy to see that

$$g'(0^+) = (m+1)\int_\Omega (-v) H_m(u).$$

To compute the left-derivative observe that

$$\begin{aligned} &\frac{1}{t}\left(\int_\Omega (-P(u+tv))(dd^c P(u+tv))^m \wedge \beta^{n-m} - \int_\Omega (-u)(dd^c u)^m \wedge \beta^{n-m}\right) \\ &= \sum_{k=0}^m \int_\Omega \frac{u-P(u+tv)}{t}(dd^c u)^k \wedge (dd^c P(u+tv))^{m-k} \wedge \beta^{n-m}. \end{aligned}$$

It suffices to apply Lemma 4.8. □

4.2. **Resolution.** In this section we use the variational formula established from above to solve the equation $H_m(u) = \mu$ in Cegrell's classes, where μ is a positive Radon measure. The following Lemma is important for the sequel.

Lemma 4.10. *Let μ be a positive Radon measure such that $\mu(\Omega) < +\infty$. Suppose that there exists $A > 0$ such that*

$$\int_\Omega (-\varphi)^2\, d\mu \leq A\, e_1(\varphi)^{\frac{2}{m+1}} \text{ for every } \varphi \in \mathcal{E}_m^1(\Omega). \tag{4.4}$$

Then $\mu \in \mathcal{M}_1$. Moreover, if $\{v_j\} \subset \mathcal{E}_m^1(\Omega)$ is a sequence such that $\sup_j e_1(v_j) < +\infty$, then we can extract a subsequence $\{v_{j_k}\}$ such that

$$\int_\Omega v_{j_k} d\mu \to \int_\Omega v d\mu.$$

Finally there exists a unique function $u \in \mathcal{E}_m^1(\Omega)$ such that $H_m(u) = \mu$.

Proof. Fix $\varphi \in \mathcal{E}_m^1(\Omega)$. From (4.4) we can find a constant $A > 0$ such that

$$\begin{aligned}\int_\Omega (-\varphi)\, d\mu \leq \left(\int_\Omega (-\varphi)^2\, d\mu\right)^{1/2} \mu(\Omega)^{1/2} &\leq A^{1/2} e_1(\varphi)^{\frac{1}{m+1}} \mu(\Omega)^{1/2} \\ &= Ce_1(\varphi)^{\frac{1}{m+1}} < +\infty\,,\end{aligned}$$

which implies that $\mu \in \mathcal{M}_1$.

Suppose now that $\{v_j\} \subset \mathcal{E}_m^1(\Omega)$ satisfies $\sup_j e_1(v_j) = C < +\infty$. The compactness of $\mathcal{E}_m^{1,C}$ (Lemma 4.1) allows us to extract a subsequence (still denoted by (v_j)) converging in the sense of distributions to $v \in \mathcal{E}_m^1(\Omega)$. The inequality (4.4) gives us

$$\sup_j \int_\Omega (-v_j)^2\, d\mu < +\infty.$$

By Lemma 4.2 we see that $\int_\Omega v_j\, d\mu \to \int_\Omega v\, d\mu$.

We now prove the last statement. Let $(u_j) \subset \mathcal{E}_m^1(\Omega)$ be such that

$$\lim_j \mathcal{F}_\mu(u_j) = \inf_{v\in\mathcal{E}_m^1(\Omega)} \mathcal{F}_\mu(v) \leq 0.$$

From the properness of the functional $\mathcal{F}_\mu$ (Lemma 4.5), we obtain $\sup_j e_1(u_j) < +\infty$. It follows from the fist part of the proof that there exists a subsequence (still denoted by (u_j)) such that u_j converges to u in $L^1_{\mathrm{loc}}(\Omega)$ and $\int_\Omega u_j d\mu \to \int_\Omega u d\mu$. From Lemma 4.3 we see that e_1 is lower semicontinuous. Thus,

$$\liminf_{j\to\infty} \mathcal{F}_\mu(u_j) = \liminf_{j\to\infty} e_1(u_j) + \lim_{j\to\infty} \int_\Omega u_j d\mu \geq e_1(u) - \|u\|_1 = \mathcal{F}_\mu(u)\,.$$

We then deduce that u is a minimum point of $\mathcal{F}_\mu$ on $\mathcal{E}_m^1(\Omega)$.

Now, let $v \in \mathcal{E}_m^0(\Omega) \cap \mathcal{C}(\Omega)$. The function $g(t) := \frac{1}{m+1} e_1(P(u+tv)) + \int_\Omega (u+tv) d\mu$ is differentiable 0 and

$$g'(0) = -\int_\Omega v H_m(u) + \int_\Omega v d\mu.$$

Since $P(u+tv) \leq u+tv$, we have $g(t) \geq \mathcal{F}_\mu(e_1(P(u+tv)) \geq \mathcal{F}_\mu(u) = g(0),\ \forall t \in \mathbb{R}$. Hence $g'(0) = 0$ which gives

$$\int_\Omega v H_m(u) = \int_\Omega v d\mu.$$

□

Theorem 4.11. *If $\mu \in \mathcal{M}_1$ then there exists a unique $u \in \mathcal{E}_m^1(\Omega)$ such that $H_m(u) = \mu$.*

Proof. The uniqueness follows from the comparison principle.

We prove the existence. Suppose first that μ has compact support $K \Subset \Omega$, and let $h_K := h^*_{m,K,\Omega}$ denote the m-extremal function of K with respect to Ω. Set

$$\mathcal{M} = \left\{\nu \geq 0 \ /\ \mathrm{supp}\nu \subset K,\ \int_\Omega (-\varphi)^2 d\nu \leq Ce_1(\varphi)^{\frac{2}{m+1}} \text{ for every } \varphi \in \mathcal{E}_m^1(\Omega)\right\},$$

where C is a fixed constant such that $C > 2e_1(h_K)^{\frac{n-1}{n+1}}$. For each compact $L \subset K$, we have $h_K \leq h_L$. We deduce that $e_1(h_L) \leq e_1(h_K)$. Therefore,

$$\begin{aligned}\int_\Omega (-\varphi)^2 H_m(h_L) &\leq 2\|h_L\| \int_\Omega (-\varphi)(dd^c\varphi) \wedge (dd^c h_L)^{m-1} \wedge \omega^{n-m} \\ &\leq 2\left(\int_\Omega (-\varphi) H_m(\varphi)\right)^{\frac{2}{m+1}} \left(\int_\Omega (-h_L) H_m(h_L)\right)^{\frac{m-1}{m+1}} \\ &\leq Ce_1(\varphi)^{\frac{2}{m+1}},\end{aligned}$$

for every $\varphi \in \mathcal{E}^1_m(\Omega)$. This implies that $H_m(h_L) \in \mathcal{M}$ for every compact $L \subset K$.

Put $T = \sup\{\nu(\Omega) \ / \ \nu \in \mathcal{M}\}$. We claim that $T < +\infty$. In fact, since Ω is m-hyperconvex, there exists $h \in \mathcal{SH}^-_m(\Omega) \cap \mathcal{C}(\bar{\Omega})$ such that

$$K \Subset \{h < -1\} \Subset \Omega.$$

For each $\nu \in \mathcal{M}$, we have

$$\nu(K) \leq \int_K (-h) d\nu \leq C.e_1(h)^{\frac{2}{m+1}},$$

from which the claim follows.

Fix $\nu_0 \in \mathcal{M}$ such that $\nu_0(\Omega) > 0$. Define

$$\mathcal{M}' = \Bigg\{ \nu \geq 0 \ / \ \nu(\Omega) = 1, \ \mathrm{supp}\nu \subset K, \\ \int_\Omega (-\varphi)^2 d\nu \leq \left(\frac{C}{T} + \frac{C}{\nu_0(\Omega)}\right) e_1(\varphi)^{\frac{2}{m+1}} \text{ for every } \varphi \in \mathcal{E}^1_m(\Omega) \Bigg\},$$

Then, for each $\nu \in \mathcal{M}$ and $\varphi \in \mathcal{E}^1_m(\Omega)$,

$$\begin{aligned}\int_\Omega (-\varphi)^2 \frac{(T-\nu(\Omega))d\nu_0 + \nu_0(\Omega)d\nu}{T\nu_0(\Omega)} &\leq \frac{T-\nu(\Omega)}{T\nu_0(\Omega)} \int_\Omega (-\varphi)^2 d\nu_0 + \frac{1}{T}\int_\Omega (-\varphi)^2 d\nu \\ &\leq \left(C\frac{T-\nu(\Omega)}{T\nu_0(\Omega)} + \frac{C}{T}\right) e_1(\varphi)^{\frac{2}{m+1}} \\ &\leq \left(\frac{C}{\nu_0(\Omega)} + \frac{C}{T}\right) e_1(\varphi)^{\frac{2}{m+1}}.\end{aligned}$$

From this we infer that

$$\frac{(T-\nu(\Omega))\nu_0 + \nu_0(\Omega)\nu)}{T\nu_0(\Omega)} \in \mathcal{M}', \ \text{for every } \nu \in \mathcal{M}.$$

We conclude that $\mathcal{M}'$ is (non empty) convex and weakly compact in the space of probability measures. It follows from a generalized Radon-Nykodim Theorem [25] that there exists a positive measure $\nu \in \mathcal{M}'$ and a positive function $f \in L^1(\nu)$ such that $\mu = f\,d\nu + \nu_s$, where ν_s is orthogonal to $\mathcal{M}'$. Observe also that every measures orthogonal to $\mathcal{M}'$ is supported in some m-polar set since $H_m(h_L) \in \mathcal{M}$ for each $L \Subset K$. We then deduce that $\nu_s \equiv 0$ since μ does not charge m-polar sets.

From Lemma 4.10, for each $\lambda \in \mathcal{M}'$, there exists a unique function $u \in \mathcal{E}^1_m(\Omega)$ such that $(dd^c u)^n = \lambda$. For each $j \in \mathbb{N}$ set $\mu_j = \min(f, j)\nu$. Then μ_j satisfies (4.4) since $\mu_j \leq j.\nu$. Therefore, there exists $u_j \in \mathcal{E}^1_m(\Omega)$ such that $H_m(u_j) = \mu_j$. It is clear that $\{u_j\}$ decrease to a function $u \in \mathcal{E}^1_m(\Omega)$ which verifies $H_m(u) = \mu$.

It remains to treat the case when μ does not have compact support. Let $\{K_j\}$ be an exhaustion sequence of compact subset of Ω and consider $\mu_j = \chi_{K_j} d\mu$. Observe also that (u_j) decrease to $u \in \mathcal{SH}_m^-(\Omega)$. It suffices to prove that $\sup_j e_1(u_j) < +\infty$. Since $\mu \in \mathcal{M}_1$, we have

$$e_1(u_j) = \int_\Omega (-u_j) H_m(u_j) = \int_{K_j} (-u_j) d\mu \leq \int_\Omega (-u_j) d\mu \leq A e_1(u_j)^{\frac{1}{m+1}},$$

This implies that $e_1(u_j)$ is uniformly bounded, from which the result follows. □

Lemma 4.12. *Let μ be a positive Radon measure having finite mass $\mu(\Omega) < +\infty$. Assume that $\mu \leq H_m(\psi)$, where ψ is a a bounded m-sh function in Ω. Then there exists a unique function $\varphi \in \mathcal{F}_m^1(\Omega)$ such that $\mu = H_m(\varphi)$.*

Proof. Without loss of generality, we can assume that $-1 \leq \psi \leq 0$. Consider $h_j = \max(\psi, jh)$, where $h \in \mathcal{E}_m^0(\Omega)$ is an exhaustion function of Ω. Let $A_j := \{z \in \Omega \ / \ jh < -1\}$. From Theorem 4.11, there exists $(\varphi_j)_j \subset \mathcal{E}_m^0(\Omega)$ such that $H_m(\varphi_j) = 1\!\!1_{A_j}\mu, \ \forall j$. Thus,

$$0 \geq \varphi_j \geq h_j \geq \psi, \text{ and } \varphi_j \downarrow \varphi \in \mathcal{F}_m^1(\Omega).$$

□

Now we prove a decomposition theorem of type Cegrell.

Theorem 4.13. *Let μ be a positive measure in Ω which does not charge m-polar sets. Then there exists $\varphi \in \mathcal{E}_m^0(\Omega)$ and $0 \leq f \in L^1_{\text{loc}}(H_m(\varphi))$ such that $\mu = f.H_m(\varphi)$.*

Proof. We first assume that μ has compact support. By applying Theorem 4.11 we can find $u \in \mathcal{E}_m^1(\Omega)$ and $0 \leq f \in L^1(H_m(u))$ such that $\mu = f.H_m(u)$, and $\text{supp}(H_m(u)) \Subset \Omega$. Consider

$$\psi = (-u)^{-1} \in \mathcal{SH}_m(\Omega) \cap L^\infty_{\text{loc}}(\Omega).$$

Then $(-u)^{-2m} H_m(u) \leq H_m(\psi)$. Since $H_m(u)$ has compact support in Ω, we can modify ψ in a neighborhood of $\partial\Omega$ such that $\psi \in \mathcal{E}_m^0(\Omega)$. It follows from Lemma 4.12 that

$$(-u)^{-2m} H_m(u) = H_m(\varphi), \ \varphi \in \mathcal{E}_m^0(\Omega).$$

This gives us $\mu = f(-u)^{2m}.H_m(\varphi)$.

It remains to consider the case μ does not have compact support. Let (K_j) be an exhaustive sequence of compact subsets of Ω. From previous arguments there exists $u_j \in \mathcal{E}_m^0(\Omega)$ and $f_j \in L^1(H_m(u_j))$ such that $1\!\!1_{K_j}\mu = f_j H_m(u_j)$. Take a sequence of positive numbers (a_j) verifying $\varphi := \sum_{j=1}^\infty a_j u_j \in \mathcal{E}_m^0(\Omega)$. The measure μ is absolutely continuous with respect to $H_m(\varphi)$. Thus,

$$\mu = g H_m(\varphi) \text{ et } g \in L^1_{\text{loc}}(H_m(\varphi)).$$

□

Proposition 4.14. *Let μ be a positive Radon measure on Ω and $p > 0$. Then $\mathcal{E}_m^p(\Omega) \subset L^p(\mu)$ if and only if there exists a positive constant $C > 0$ such that*

$$\int_\Omega (-u)^p d\mu \leq C e_p(u)^{\frac{p}{m+p}}, \ \forall u \in \mathcal{E}_m^p(\Omega).$$

Proof. One implication is evident. For the other, suppose that μ satisfies $\mathcal{E}_m^p(\Omega) \subset L^p(\mu)$ and there exists a sequence $(u_j) \subset \mathcal{E}_m^p(\Omega)$ such that

$$\int_\Omega (-u_j)^p d\mu \geq 4^{jp} e_p(u_j)^{\frac{p}{m+p}}.$$

For simplicity we can assume that $e_p(u_j) = 1, \forall j$. By Corollary 3.11, $v = \sum_{j=1}^\infty 2^{-j} v_j$ belongs to $\mathcal{E}_m^p(\Omega)$. But

$$\int_\Omega (-v)^p d\mu \geq \int_\Omega (-2^{-j} v_j)^p d\mu \geq 2^{jp} \to +\infty,$$

which contradicts $\mathcal{E}_m^p(\Omega) \not\subset L^p(\mu)$. $\square$

Remark 4.15. If $u, v \in \mathcal{E}_m^p(\Omega)$ and $(u_j), (v_j) \subset \mathcal{E}_m^0(\Omega)$ decrease to u, v respectively, then by Lemma 3.8 and Proposition 3.10 we have

$$\int_\Omega (-u)^p H_m(v) \leq \liminf_j \int_\Omega (-u_j)^p H_m(v_j) < +\infty.$$

Thus, $\mathcal{E}_m^p(\Omega) \subset L^p(H_m(v))$ and by Proposition 4.14 there exists $C_v > 0$ such that

$$\int_\Omega (-u)^p H_m(v) \leq C_v e_p(u)^{\frac{p}{p+m}}, \ \forall u \in \mathcal{E}_m^p(\Omega).$$

It is not clear how to obtain this inequality directly by using Hölder inequality.

Theorem 4.16. *Let μ be a positive Radon measure on Ω such that $\mathcal{E}_m^p(\Omega) \subset L^p(\mu), p > 0$. Then there exists a unique $\varphi \in \mathcal{E}_m^p(\Omega)$ such that $H_m(\varphi) = \mu$.*

Proof. The uniqueness follows from the comparison principle. Let us prove the existence result. Since μ does not charge m-polar sets, applying the decomposition theorem (Theorem 4.13) we get

$$\mu = f H_m(u), \quad u \in \mathcal{E}_m^0(\Omega), \ 0 \leq f \in L^1_{\rm loc}(H_m(u)).$$

For each j, use Lemma 4.12 to find $\varphi_j \in \mathcal{E}_m^0(\Omega)$ such that

$$H_m(\varphi_j) = \min(f, j) H_m(u).$$

By Proposition 4.14, $\sup_j e_p(\varphi_j) < +\infty$. Thus, the comparison principle gives us that $\varphi_j \downarrow \varphi \in \mathcal{E}_m^p(\Omega)$ which solves $H_m(\varphi) = \mu$. $\square$

Theorem 4.17. *Let μ be a positive Radon measure on Ω with finite total mass $\mu(\Omega) < +\infty$. If μ does not charge m-polar sets then there exists a unique $\varphi \in \mathcal{F}_m(\Omega)$ such that $H_m(\varphi) = \mu$.*

Proof. We first prove the existence result. Since μ does not charge m-polar sets the decomposition theorem yields

$$\mu = f H_m(u), \quad u \in \mathcal{E}_m^0(\Omega), \ 0 \leq f \in L^1_{\rm loc}(H_m(u)).$$

For each j use Lemma 4.12 to find $\varphi_j \in \mathcal{E}_m^0(\Omega)$ such that

$$H_m(\varphi_j) = \min(f, j) H_m(u).$$

Besides, $\sup_j \int_\Omega H_m(\varphi_j) \leq \mu(\Omega) < +\infty$. Thus, $\varphi_j \downarrow \varphi \in \mathcal{F}_m(\Omega)$ in view of the comparison principle. The limit function φ solves $H_m(\varphi) = \mu$ as required.

To prove the uniqueness we follow the lines in [6, Lemma 5.14]. Assume that $\psi \in \mathcal{F}_m(\Omega)$ solves $H_m(\psi) = \mu$. We are to prove that $\varphi = \psi$. Let (K_j) be an exhaustive sequence of compact subsets of Ω such that $h_j = h_{m,K_j,\Omega}$ is continuous.

For each j, the function $\psi_j := \max(\psi, j.h_j)$ belongs to $\mathcal{E}_m^0(\Omega)$, and $\psi_j \downarrow \psi$. Set $d_j := \frac{\psi_j}{j} - h_j = \max(\frac{\psi}{j} - h_j, 0)$. Then $d_j \leq 1\!\!1_{\{\psi > j.h_j\}}$ and $1 - d_j \downarrow 0$. For $s > j$, by the comparison principle we get

$$\begin{aligned} 0 \leq d_j H_m(\max(\psi, s.h_j)) &\leq 1\!\!1_{\{\psi > j.h_j\}} H_m(\max(\psi, s.h_j)) \\ &= 1\!\!1_{\{\psi > j.h_j\}} H_m(\max(\psi, j.h_j)) \\ &= H_m(\max(\psi, s.h_j)). \end{aligned}$$

Letting s tend to $+\infty$ and using Corollary 3.16 we get

$$d_j.H_m(\psi) \leq 1\!\!1_{\{\psi > j.h_j\}} H_m(\max(\psi, j.h_j)) \leq H_m(\psi). \tag{4.5}$$

Recall that from the first part we have

$$\mu = f H_m(u), \quad u \in \mathcal{E}_m^0(\Omega),\ 0 \leq f \in L^1_{\text{loc}}(H_m(u)).$$

and $H_m(\varphi_p) = \min(f, p) H_m(u)$ for each $p \in \mathbb{N}$. For each p, j we can find $v_j^p \in \mathcal{E}_m^0(\Omega)$ such that

$$H_m(v_j^p) = (1 - d_j) H_m(\varphi_p).$$

Using (4.5) we get

$$\begin{aligned} H_m(\varphi_p) &= d_j.H_m(\varphi_p) + (1 - d_j) H_m(\varphi_p) \\ &\leq d_j H_m(\psi) + (1 - d_j) H_m(\varphi_p) \\ &\leq 1\!\!1_{\{\psi > j h_j\}} H_m(\psi_j) + H_m(v_j^p) \\ &\leq H_m(\psi_j) + H_m(v_j^p). \end{aligned} \tag{4.6}$$

This couped with the comparison principle yield $\varphi_p \geq v_j^p + \psi_j$. Letting $p \to +\infty$ we obtain $\varphi \geq v_j + \psi_j$, where $v_j \in \mathcal{F}_m(\Omega)$ solves $H_m(v_j) = (1 - d_j) H_m(\varphi)$. Since $H_m(\varphi)$ does not charge m-polar sets, by monotone convergence theorem the total mass of $H_m(v_j)$ goes to 0 as $j \to +\infty$. This implies that v_j increases to 0 and hence $\varphi \geq \psi$.

Now, we prove that $\varphi \leq \psi$. Let $\psi_j, t_j \in \mathcal{E}_m^0(\Omega)$ such that $H_m(w_j) = d_j H_m(\psi_j)$ and $H_m(t_j) = (1 - d_j) H_m(\psi_j)$. Since $H_m(\varphi_p)$ increases to $H_m(\varphi)$, the comparison principle can be applied for φ and w_j which implies that $w_j \geq \varphi$. But, applying again the comparison principle for $t_j + w_j$ and ψ_j we get $t_j + w_j \leq \psi_j$. Furthermore, the total mass of $H_m(t_j)$ can be estimated as follows

$$\begin{aligned} \int_\Omega H_m(t_j) &= \int_\Omega H_m(\psi_j) - \int_\Omega d_j H_m(\psi_j) \\ &\leq \int_\Omega H_m(\psi) - \int_\Omega d_j^2 H_m(\psi) \\ &\leq 2 \int_\Omega (1 - d_j) H_m(\psi) \to 0. \end{aligned}$$

This implies that t_j converges in m-capacity to 0. Indeed, for every $\epsilon > 0$ and m-subharmonic function $-1 \leq \theta \leq 0$, by the comparison principle we have

$$\begin{aligned} \epsilon^m \int_{\{t_j \leq -\epsilon\}} H_m(\theta) &\leq \int_{\{t_j \leq -\epsilon\theta\}} H_m(\epsilon\theta) \\ &\leq \int_{\{t_j \leq -\epsilon\theta\}} H_m(t_j) \leq \int_\Omega H_m(t_j) \to 0. \end{aligned}$$

Thus, we can deduce that $\varphi \leq \psi$ which implies the equality. $\square$

Remark 4.18. We prove in the above uniqueness theorem that every $\psi \in \mathcal{F}_m^a(\Omega)$ (functions in $\mathcal{F}_m(\Omega)$ whose Hessian measure does not charge m-polar sets) can be approximated from above by sequence $\psi_j \in \mathcal{E}_m^0(\Omega)$ such that $H_m(\psi_j)$ increases to $H_m(\psi)$. This type of convergence is strong enough to prove the comparison principle for the class $\mathcal{F}_m^a(\Omega)$.

Theorem 4.19. *The comparison principle is valid for functions in* $\mathcal{F}_m^a(\Omega)$.

Proposition 4.20. *Let* $u, v \in \mathcal{E}_m^p(\Omega),\ p > 0$. *There exist two sequences* $(u_j), (v_j) \subset \mathcal{E}_m^0(\Omega)$ *decreasing to* u, v *respectively such that*

$$\lim_{j \to +\infty} \int_\Omega (-u_j)^p H_m(v_j) = \int_\Omega (-u)^p H_m(v).$$

In particular, if $\varphi \in \mathcal{E}_m^p(\Omega)$ *then there exists* $(\varphi_j) \subset \mathcal{E}_m^0(\Omega)$ *decreasing to* φ *such that*

$$e_p(\varphi_j) \to e_p(\varphi).$$

Proof. Let (u_j) be a sequence in $\mathcal{E}_m^0(\Omega)$ decreasing to u such that $\sup_j \int_\Omega (-u_j)^p H_m(u_j) < +\infty$. Since $H_m(v)$ vanishes on m-polar sets Theorem 4.13 gives

$$H_m(v) = f H_m(\psi), \quad \psi \in \mathcal{E}_m^0(\Omega),\ 0 \le f \in L^1_{\text{loc}}(H_m(\psi)).$$

For each j, use Lemma 4.12 to find $v_j \in \mathcal{E}_m^0(\Omega)$ such that

$$H_m(v_j) = \min(f, j) H_m(\psi).$$

By the comparison principle $v_j \downarrow \varphi \in \mathcal{E}_m^p(\Omega)$ which solves $H_m(\varphi) = H_m(v)$. It implies that $\varphi \equiv v$. We the have

$$\int_\Omega (-u)^p H_m(v) = \lim_j \int_\Omega (-u_j)^p \min(f, j) H_m(\psi) = \lim_j \int_\Omega (-u_j)^p H_m(v_j).$$

□

4.3. **Examples.**

Lemma 4.21. *If* $\varphi \in \mathcal{E}_m^p(\Omega)$, $p > 0$ *then*

$$Cap_m(\varphi < -t) \le C.e_p(\varphi).\frac{1}{t^{m+p}},$$

where $C > 0$ *is a constant depending only on* m.

Proof. Without loss of generality we can assume that $\varphi \in \mathcal{E}_m^0(\Omega)$. Fix $u \in \mathcal{P}_m^-(\Omega)$ such that $-1 \le u \le 0$. Observe that, for any $t > 0$,

$$(\varphi < -2t) \subset (\varphi < tu - t) \subset (\varphi < -t).$$

By the comparison principle (Corollary 2.18) we have

$$\begin{aligned}\int_{\varphi < -2t} H_m(u) &\le \frac{1}{t^m} \int_{\varphi < tu - t} H_m(tu - t) \le \frac{1}{t^m} \int_{\varphi < tu - t} H_m(\varphi) \\ &\le \frac{1}{t^m} \int_{\varphi < -t} H_m(\varphi) \le \frac{1}{t^{m+p}} \int_\Omega (-\varphi)^p H_m(\varphi).\end{aligned}$$

□

Proposition 4.22. *Let $\mu = fdV$, where $0 \leq f \in L^p(\Omega, dV)$, $\frac{n}{m} > p > 1$. Then*

$$\mu = H_m(\varphi),\ \varphi \in \mathcal{F}_m^q(\Omega),\ \forall q < \frac{nm(p-1)}{n-mp}.$$

Proof. Fix $0 < r < \frac{n}{n-m}$. By Hölder's inequality, Proposition 2.1 in [7], there exists $C > 0$ depending only on $p, r \int_\Omega f^p dV$ such that

$$\mu(K) \leq CVol(K)^{\frac{p-1}{p}} \leq C.Cap_m(K)^{\frac{r(p-1)}{p}}. \tag{4.7}$$

Take $0 < q < \frac{nm(p-1)}{n-mp}$ and $u \in \mathcal{E}_m^q(\Omega)$. By Theorem 4.16 it suffices to show that $u \in L^q(\mu)$ which is, in turn, equivalent to showing that

$$\int_1^{+\infty} \mu(u < -t^{1/q})dt < +\infty.$$

The latter follows easily from (4.7) and Lemma , which completes the proof. □

The exponent $q(p) = \frac{nm(p-1)}{n-mp}$ is sharp in view of the following example.

Example 4.23. Consider $\varphi_\alpha = 1 - \|z\|^{-2\alpha}$, where α is a constant in $(0, \frac{n-m}{m})$. An easy computation shows that $\varphi_\alpha \in \mathcal{F}_m(\Omega)$ and

$$H_m(\varphi_\alpha) = C.\|z\|^{-2m(\alpha+1)}dV = f_\alpha dV.$$

Then

$$\varphi_\alpha \in \mathcal{F}_m^q(\Omega) \Longleftrightarrow q < \frac{n-m}{\alpha} - m,$$

while

$$f_\alpha \in L^p(\Omega, dV) \Longleftrightarrow p < \frac{n}{m(\alpha+1)}.$$

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LU Hoang Chinh
Université Paul Sabatier
Institut de Mathématiques de Toulouse
118 Route de Narbonne
31062 Toulouse
lu@math.univ-toulouse.fr.